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CONSTRUCTING INTEGRO-DIFFERENTIAL EQUATION FOR THE GERBER-SHIU FUNCTION IN ERLANG(*n*) INSURANCE RISK MODEL WITH CONSTANT INTEREST RATE

In this paper, an Erlang(n) insurance risk model with constant interest force is investigated. An integro-differential equation which Gerber-Shiu function satisfies is derived.

Keywords: Gerber-Shiu function, Erlang(*n*) process, interest rate, integro-differential equation

1. Introduction. The study of the expected discounted penalty function is of considerable iterest in the risk theory. The expected discounted penalty function has been defined by Gerber and Shiu [1] in the classical Poisson risk model and the defective renewal equation satisfied by the expected value has been derived. The penalty function has been investigated widely in the classical Poisson risk model and Erlang(2) risk model with and without interest (see, for example, [2, 3, 4]). An integro-differential equation for the expected discounted penalty function of Erlang(n) no-interest risk model has been established by Xing and Wu [5].

It is known that a large portion of the surplus of the insurance company comes from investment income, and interest factor must affect the management of the company. So it is necessary to discuss the ruin problem by taking into account interest factor. Gaoqin, et al [6] investigated the expected discounted penalty function of Erlang(2) risk model with a constant interest rate and obtained an integro-differential equation of the expected value. In this paper, we consider the expected discounted penalty function of Erlang(n) risk model with a constant interest rate. An integro-differential equation (see, for example, Svetova and Semenova [7]) for the expected discounted penalty function is obtained.

2. The model. Consider the insurance risk process

$$U_\delta(t) = ue^{\delta t} + c\int_0^t e^{\delta(t-y)}dy - \sum_{i=1}^{N(t)} X_i e^{\delta\left(t-\sum_{j=1}^{t} T_j\right)}, \tag{2.1}$$

where $u = U_\delta(0) \geq 0$ is the initial capital of the insurance company, $c > 0$ is the premium rate, δ is the constant interest force, $\{X_i, i \geq 1\}$ denotes the sequence of independent and identically distributed (i.i.d.) non-negative successive claims, and $N(t), (t \geq 0)$ denotes the number of claims up to time t, which is a counting process independent of $\{X_i, i \geq 1\}$.

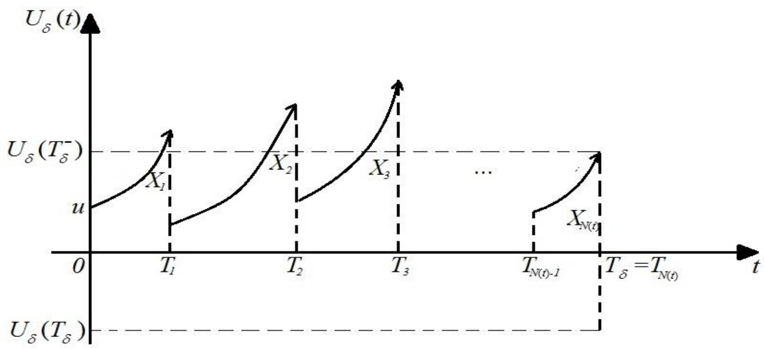

Figure 2.1. A trajectory of the process $U_\delta(t)$

If $N(t)$ is a renewal process, that is, the times $T_i, i \geq 1$, elapsed between successive claims are i.i.d., and there is no interest, that is, $\delta = 0$, the model above is the renewal risk model introduced by Sparre Anderson (see, for exapmle, [8, 9]):

$$U(t)=u+ct-\sum_{i=1}^{N(t)}X_i. \tag{2.2}$$

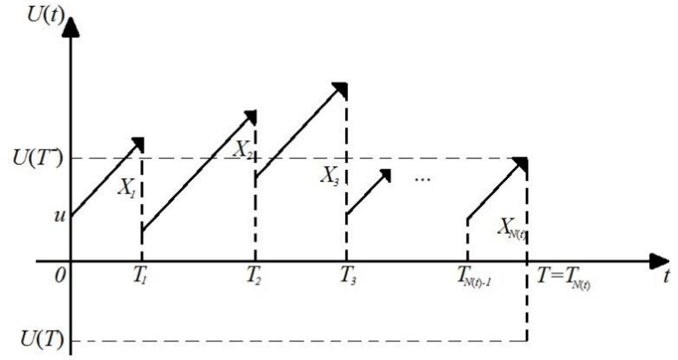

Figure 2.2. A trajectory of the process $U(t)$.

In this paper, we consider the Erlang(n) risk model with a constant interest rate, where T_1 has an Erlang(n) probability density function (p.d.f.) $\gamma_n(t)$ with scale parameter $\beta>0$:

$$\gamma_n(t)=\frac{\beta^n}{(n-1)!}t^{n-1}e^{-\beta t}, \qquad t>0. \tag{2.3}$$

Now define the time of ruin by $T_\delta=\inf\{t: U_\delta(t)<0\}$, and $T_\delta=\infty$ if $U_\delta(t)\geq 0$ for all $t>0$. Then the ruin probability is defined as $\Psi_{n;\delta}(u)=\mathrm{P}\{T_\delta<\infty|U_\delta(0)=u\}$. Let $U_\delta(T_\delta^-)$ and $|U_\delta(T_\delta)|$ denote the surplus immediately prior to ruin and deficit at ruin when ruin occurs, respectively. We consider the expected discounted penalty function of the surplus immediately prior to ruin and the deficit at ruin when ruin occurs as a function of the initial surplus u, namely

$$\Phi_{n;\delta,\alpha}(u)=\mathrm{E}\left\{e^{-\alpha T_\delta}\omega(U_\delta(T_\delta^-),|U_\delta(T_\delta)|)\mathbb{I}_{(T_\delta<\infty)}\middle|U_\delta(0)=u\right\}, \tag{2.4}$$

where $\alpha\geq 0$, $\mathbb{I}_B$ is the indicator function of a set B, and $\omega(x_1,x_2)$, $0<x_1,x_2<0$, is a non-negative function. Many properties of the surplus process may be obtained from this general function. In the case of $\alpha=0$, choosing different forms of the function $\omega(x_1,x_2)$, we obtain different information relating to the deficit at ruin and the surplus prior to ruin. For example, $\Phi_{n;\delta,\alpha}(u)$ will represent the ν-order moment of the deficit at ruin (or the surplus prior to ruin) if we specially choose $\omega(x_1,x_2)=x_1^\nu$ (or x_1^ν), represent their joint c.d.f. if $\omega(x_1,x_2)=\mathbb{I}_{(x_1\leq x,x_2\leq y)}$, represent c.d.f. of the deficit at ruin if $\omega(x_1,x_2)=\mathbb{I}_{(x_2\leq x)}$ and so on. When $\omega(x_1,x_2)\equiv 1$, $\Phi_{n;\delta,\alpha}(u)$ reduces to $\Psi_{n;\delta}(u)$.

Throughout this paper, we assume that the claim sizes, X_i, $i\geq 1$, are i.i.d. with a common c.d.f. F supported on $[0,\infty)$, a finite mean μ, and p.d.f. $f(x)$. It is always assumed that the safety loading condition

$$\rho=\frac{c\mathrm{E}\{T_1\}-\mu}{\mu}=\frac{cn/\beta-\mu}{\mu}>0 \tag{2.5}$$

holds.

3. Main results. Using the renewal property of the surplus process, an integro-differential equation satisfied by the expected discounted penalty function $\Phi_{n;\delta,\alpha}(u)$ is derived. The main idea is based on the fact that we consider a delayed renewal process, that is, the time T_1, elapsed before the first claim, has an Erlang(k) ($k\leq n$) probability density function with scale parameter $\beta>0$, and the interoccurrence times T_i, $i>1$, after the first claim, have the same p.d.f. Erlang(n) with the same scale parameter β, and are independent of each other. If the first claim has occurred at time 0 and the ruin has not occurred, the risk model is also an Erlang(n) risk model.

We denote by $\phi_{k;\delta,\alpha}\left(\phi_{n;\delta,\alpha}=\Phi_{n;\delta,\alpha}(u)\right)$ the penalty function corresponding to the delayed renewal risk model.

Theorem 3.1. Consider the Erlang(n) risk model with the relative safety loading condition (2.5) under constant interest force. If

$$\int_0^\infty\int_0^\infty \omega(x_1,x_2)f(x_1+x_2)dx_1\,dx_2<\infty, \tag{3.1}$$

Then $\Phi_{n;\delta,\alpha}(u)$ satisfies the following integro-differential equation:

$$\beta^n\int_0^u \Phi_{n;\delta,\alpha}(u-x)f(x)dx+\beta^n\int_u^\infty \omega(u,x-u)f(x)dx=$$

$$=[(\alpha+\beta)I-(\delta u+c)D]^n\Phi_{n;\delta,\alpha}(u), \tag{3.2}$$

where I and D denote the identity operator and differentiation operator, respectively, and

$$[(\alpha+\beta)I-(\delta u+c)D]^n\Phi_{n;\delta,\alpha}(u)=$$

$$=\sum_{m=0}^{n}C_n^m(\alpha+\beta)^{n-m}(-1)^m\sum_{l=0}^{m}V_m^l\delta^{m-l}\sum_{r=0}^{l}C_l^r\delta^r c^{l-r}u^r\,\Phi_{n;\delta,\alpha}^{(l)}(l),$$

$$C_n^m=\frac{n!}{m!\,(n-m)!},$$

$$V_m^0=0,\qquad m\ge 1,\qquad V_0^0=1,$$

$$V_m^1=V_m^m=1,\qquad m\ge 1,$$

$$V_m^l=lV_m^{l-1}+V_{m-1}^{l-1},\qquad 2\le l\le m-1,\qquad m\ge 3,$$

$$\Phi_{n;\delta,\alpha}^{(l)}(u)=\frac{d^l}{du^l}\Phi_{n;\delta,\alpha}(u),\qquad l\ge 1,\qquad \Phi_{n;\delta,\alpha}^{(0)}(u)=\Phi_{n;\delta,\alpha}(u).$$

Proof. We have (see, for example, [5, 6])

$$\phi_{k;\delta,\alpha}(u)=\mathrm{E}\{e^{-\alpha T_\delta}\omega(U_\delta(T_\delta^-),|U_\delta(T_\delta)|)\mathbb{I}_{(T_\delta<\infty)}|U_\delta(0)=u\}=$$

$$=\mathrm{E}\{e^{-\alpha T_\delta}\omega(U_\delta(T_\delta^-),|U_\delta(T_\delta)|)\mathbb{I}_{(T_\delta=T_1)}|U_\delta(0)=u\}+$$

$$+\mathrm{E}\{e^{-\alpha T_\delta}\omega(U_\delta(T_\delta^-),|U_\delta(T_\delta)|)\mathbb{I}_{(T_\delta>T_1)}|U_\delta(0)=u\}=$$

$$=\int_0^\infty e^{-\alpha t}\gamma_k(t)\int_{ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}}^{\infty}\omega\left(ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta},x-ue^{\delta t}-c\frac{e^{\delta t}-1}{\delta}\right)f(x)dx\,dt+$$

$$+\int_0^\infty e^{-\alpha t}\gamma_k(t)\int_0^{ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}}\Phi_{n;\delta,\alpha}\left(ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}-x\right)f(x)dx\,dt$$

where, the last equation follows from the fact that once the first claim causes the ruin, the surplus before ruin and the deficit at ruin will be $ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}$ and $ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}-x$, respectively.

Substitution of $z=ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}$ into the above equation yields that

$$\phi_{k;\delta,\alpha}(u)=\int_u^\infty \frac{1}{\delta z+c}B_k\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)\int_z^\infty \omega(z,x-z)f(x)dx\,dz+$$
$$+\int_u^\infty \frac{1}{\delta z+c}B_k\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)\int_0^z \Phi_{n;\delta,\alpha}(z-x)f(x)dx\,dz, \qquad (3.3)$$

where $B_k(t)=e^{-\alpha t}\gamma_k(t),\ k\geq 1$ and $B_k'(t)=\beta B_{k-1}(t)-(\alpha+\beta)B_k(t)$ for $k\geq 2$.

Differentiating the two sides of equation (3.3) with respect to u, we obtain that for $k\geq 2$

$$(\delta u+c)\phi'_{k;\delta,\alpha}(u)=$$
$$=-\int_u^\infty \frac{1}{\delta z+c}\left[\beta B_{k-1}\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)-(\alpha+\beta)B_k\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)\right]\int_z^\infty \omega(z,x-z)f(x)dx\,dz-$$
$$-\int_u^\infty \frac{1}{\delta z+c}\left[\beta B_{k-1}\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)-(\alpha+\beta)B_k\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)\right]\int_0^z \Phi_{n;\delta,\alpha}(z-x)f(x)dx\,dz$$

So

$$\left.\begin{aligned}(\delta u+c)\phi'_{k;\delta,\alpha}(u)=-\beta\phi_{k-1;\delta,\alpha}(u)+(\alpha+\beta)\phi_{k;\delta,\alpha}(u)\\ \beta\phi_{k-1;\delta,\alpha}(u)=[(\alpha+\beta)I-(\delta u+c)D]\phi_{k;\delta,\alpha}(u)\end{aligned}\right\}, \qquad (3.4)$$

where I and D denote the identity operator and differentiation operator, respectively.

For $k=1$

$$(\delta u+c)\phi'_{1;\delta,\alpha}(u)=$$
$$=(\alpha+\beta)\int_u^\infty \frac{1}{\delta z+c}B_1\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)\int_z^\infty \omega(z,x-z)f(x)dx\,dz+$$
$$+(\alpha+\beta)\int_u^\infty \frac{1}{\delta z+c}B_1\left(\frac{1}{\delta}\ln\frac{\delta z+c}{\delta u+c}\right)\int_0^z \Phi_{n;\delta,\alpha}(z-x)f(x)dx\,dz-$$
$$-\beta\int_u^\infty \omega(u,x-u)f(x)dx-\int_0^u \Phi_{n;\delta,\alpha}(u-x)f(x)dx.$$

So

$$\beta\int_0^u \Phi_{n;\delta,\alpha}(u-x)f(x)dx+\beta\int_u^\infty \omega(u,x-u)f(x)dx=[(\alpha+\beta)I-(\delta u+c)D]\phi_{1;\delta,\alpha}(u).$$

Thus, equation (3.2) holds from successive substitution.

Theorem 3.1 is proved.

Remark 3.1. Taking $n=2$, from (3.2) we can obtain the result in Gaoqin, et al [6],taking $\delta=0$, yields us to the result obtained by Xing and Wu [5] and taking $n=2$ and $\delta=0$ the result in Cheng and Tang [10] is obtained.

Note that, if we take $\alpha=0$ and $\omega(x_1,x_2)\equiv 1$, $\Phi_{n;\delta,\alpha}(u)$ reduces to the ruin probability $\Psi_{n;\delta}(u)$. Therefore, from Theorem 3.1 can be obtained the following

Corollory 3.1. Let the condidions of Theorem 3.1 be satisfied. Additionally, let $\alpha=0$ and $\omega(x_1,x_2)\equiv 1$.Then the probability $\Psi_{n;\delta}(u)$ satisfies the following integro-differential equation:

$$\beta^n\int_0^u \Psi_{n;\delta}(u-x)f(x)dx+\beta^n\bar{F}(u)=[\beta I-(\delta u+c)D]^n\Psi_{n;\delta}(u).$$

4. Conclusion. In this paper, the expected discounted penalty function of Erlang(n) risk model with constant interest force is investigated, and an integro-differential equation which Gerber-Shiu function satisfies is derived. To this end, a delayed renewal process is considered and the properties of Erlang(k) probability density function are used. Finally, we demonstrate how we can obtain the results previously known from literature.

References

1. Gerber H.U. and Shiu E.S.W. On the time value of ruin. North American Actuarial Journal, 1998, 2(1): pp.48-78.
2. Lin X.S., Willmot G.E., Drekic S. The classical risk model with a constant dividend barrier: analysis of the Gerber-Shiu discounted penalty function. Insurance: Mathematics and Economics, 2003, 33(3): pp.551-556.
3. Dickson D.C.M., Hipp C. On the time to ruin for Erlang(2) risk processes. Insurance: Mathematics and Economics, 2001, 29(3): pp.333-344.
4. Dickson D.C.M., Hipp C. Ruin probabilities for Erlang(2) risk processes. Insurance: Mathematics and Economics, 1998, 22(3): pp.251-262.
5. Xing Y., Wu R. Moments of the time of ruin, surplus before ruin and the deficit at ruin in the Erlang(n) risk process. Acta Mathematicae Applicatae Sinica, English Series, 2006, 22(4): pp.599-606.
6. Gaoqin N., Cihua L., Lixia X. Expected discounted penalty function of Erlang(2) risk model with constant interest. Appl. Math. J. Chinese Univ. Ser. B, 2006, 21(3): pp.243-251.
7. Svetova N.Yu., Semenova E.E. Differential and integral equations. Petr SU, 2014, part 3, 56 p. (In Russian).
8. Asmussen S. Ruin Probabilities. World Scientific. 2000, p.620.
9. Aliyev R.T. Asymptotic expansions for the first four moments of Sparre Andersen risk process. Transactions of NAS Azerbaijan V.XXXVI, 3, 2016, pp.58-66.
10. Cheng Y., Tang Q. Moments of the surplus before ruin and the deficit at ruin in the Erlang(2) risk process. North American Actuarial Journal, 2003, 7(1): pp.1-12.

UOT 518.2

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Sabit faiz dərəcəli Erlanq (*n*) sığorta riski modelində Gerber-Shıufunksiyası üçün inteqro-diferensial tənliyin qurulması

Sabit faiz dərəcəli Erlanq (n) sığorta riski modeli araşdırılmış, Gerber-Shiu funksiyası üçün inteqro-diferensial tənlik qurulmuşdur.

Açar sözlər: Gerber-Shiu funksiyası, Erlanq (n) risk prosesi, faiz dərəcəsi, inteqro-diferensial tənlik

УДК 518.2

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Построение интегро-дифференциального уравнения для функции Гербер-Шью в модели страхового риска Эрланга(*n*) с постоянной процентной ставкой

Исследована модель риска Эрланга(n) с постоянной процентной ставкой. Построено интегро-дифференциальное уравнение для функции Гербер-Шью.

Ключевые слова: функция Гербер-Шью, модель риска Эрланга(n), процентная ставка, интегро-дифференциальное уравнение

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