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ON THE LAPLACE TRANSFORM OF THE GERBER-SHIU FUNCTION IN A SPARRE ANDERSEN RISK MODEL WITH CONSTANT INTEREST RATE

A Sparre Andersen insurance risk model with Erlang(n)-distributed interclaim times and a constant interest force is investigated. A n-th order differential equation for the Laplace transform of the Gerber-Shiu function is derived.

Keywords: Gerber-Shiu function, Sparre Andersen risk model, Erlang(n) risk process, Laplace transform, interest rate, integro-differential equation, n-th order differential equation

1. Introduction. The study of the expected discounted penalty function has considerable interest in the risk theory. The expected discounted penalty function has been defined by Gerber and Shiu [1] in the classical Poisson risk. The penalty function had been investigated widely in the classical Poisson risk model, Sparre Andersen risk model with Erlang(2) and Erlang(n)-distributed interclaim times (see, for example, [2-7]).

The importance of discussing the ruin problem by taking into account interest factor appears from that a large portion of the surplus of the insurance company comes from investment income, and interest factor must affect the management of the company. Gaoqin et al. [6] investigated the expected discounted penalty function of a Sparre Andersen risk model with Erlang(2)-distributed interclaim times and a constant interest rate and obtained an integro-differential equation of the expected value and using this equation they also obtained a n-th order differential equation for the Laplace transform of the Gerber-Shiu function. In study Bayramov et al. [7] an integro-differential equation for the Gerber-Shiu function in a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate was constructed. In this paper, we consider the expected discounted penalty function of a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate, too, and a n-th order differential equation for the Laplace transform of the expected value is obtained using an integro-differential equation for the Gerber-Shiu function derived in [7].

2. The model. Consider the insurance risk process

$$U_\delta(t) = ue^{\delta t} + c\int_0^t e^{\delta(t-y)}dy - \sum_{i=1}^{N(t)} X_i e^{\delta\left(t-\sum_{j=1}^{i} T_j\right)},$$

where $u = U_\delta(0) \geq 0$ is the initial capital of the insurance company, $c > 0$ is the premium rate, δ is the constant interest force, $\{X_i\}, i \geq 1$ denotes the sequence of independent and identically distributed (i.i.d.) non-negative successive claims, and $N(t), t \geq 0$ denotes the number of claims up to time t, which is a counting process independent of $\{X_i\}, i \geq 1$.

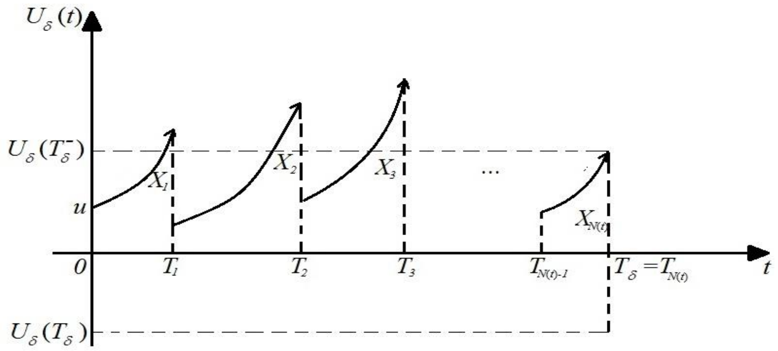

Figure 1. A trajectory of the process $U_\delta(t)$

If $N(t)$ is a renewal process, that is, the times $T_i, i \geq 1$, elapsed between successive claims are i.i.d., and there is no interest, that is, $\delta = 0$, the model above is the renewal risk model, which introduced by Sparre Andersen (see, for exapmle, Asmussen [8] and Aliyev [9]):

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i.$$

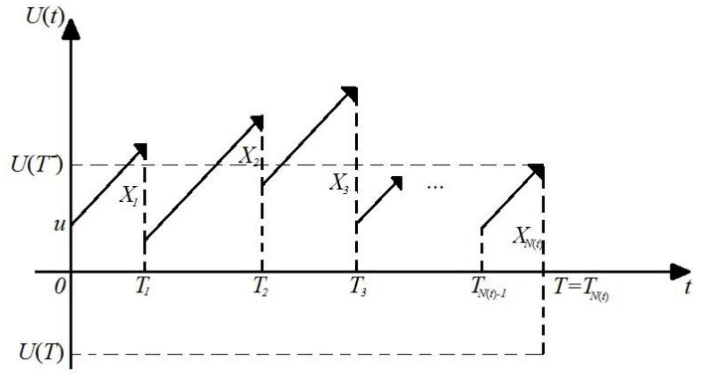

Figure 2. A trajectory of the process $U(t)$

In this paper, we consider a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate, where T_1 has an Erlang(n) probability density function (p.d.f.) $\gamma_n(t)$ with scale parameter $\beta > 0$:

$$\gamma_n(t) = \frac{\beta^n}{(n-1)!} t^{n-1} e^{-\beta t}, \qquad t > 0.$$

Now define the time of ruin by $T_\delta = \inf\{t: U_\delta(t) < 0\}$, and $T_\delta = \infty$ if $U_\delta(t) \geq 0$ for all $t > 0$. Then the ruin probability is defined as $\Psi_{n;\delta}(u) = \mathrm{P}\{T_\delta < \infty | U_\delta(0) = u\}$. Let $U_\delta(T_\delta^-)$ and $|U_\delta(T_\delta)|$ denote the surplus immediately prior to ruin and deficit at ruin when ruin occurs, respectively. We consider the expected discounted penalty function of the surplus immediately prior to ruin and the deficit at ruin when ruin occurs as a function of the initial surplus u, namely

$$\Phi_{n;\delta,\alpha}(u) = \mathrm{E}\{e^{-\alpha T_\delta} \omega(U_\delta(T_\delta^-), |U_\delta(T_\delta)|) \mathbb{I}_{(T_\delta < \infty)} \big| U_\delta(0) = u\}, \tag{2.1}$$

where $\alpha \geq 0$, $\mathbb{I}_B$ is the indicator function of a set B, and $\omega(x_1, x_2)$, $0 < x_1, x_2 < \infty$, is a non-negative function. Many properties of the surplus process may be obtained from this general function. In the case of $\alpha = 0$, choosing different forms of the function $\omega(x_1, x_2)$, we obtain different information relating to the deficit at ruin and the surplus prior to ruin. For example, $\Phi_{n;\delta,\alpha}(u)$ will represent the ν-order moment of the deficit at ruin (or the surplus prior to ruin) if we specially choose $\omega(x_1, x_2) = x_2^\nu$ (or x_1^ν), represent their joint cumulative distribution function (c.d.f.) if $\omega(x_1, x_2) = \mathbb{I}_{(x_1 \leq x, x_2 \leq y)}$, represent c.d.f. of the deficit at ruin if $\omega(x_1, x_2) = \mathbb{I}_{(x_2 \leq x)}$ and so on. When $\omega(x_1, x_2) \equiv 1$, $\Phi_{n;\delta,\alpha}(u)$ reduces to $\Psi_{n;\delta}(u)$.

Throughout this paper we suppose that the claim sizes, X_i, $i \geq 1$, are i.i.d. with a common c.d.f. F supported on $[0, \infty)$, a finite mean μ, and p.d.f. $f(x)$. It is always assumed that the safety loading condition

$$\rho = \frac{c\mathrm{E}\{T_1\} - \mu}{\mu} = \frac{cn/\beta - \mu}{\mu} > 0 \tag{2.2}$$

holds.

3. Main results. Using the renewal property of the surplus process, an integro-differential equation satisfied by the expected discounted penalty function $\Phi_{n;\delta,\alpha}(u)$ was derived in [7]. The main idea is based on the fact that a delayed renewal process is considered, that is, the time T_1, elapsed before the first claim, has an Erlang(k) ($k \le n$) p.d.f. with scale parameter $\beta > 0$, and the interoccurrence times T_i, $i > 1$, after the first claim, have the same p.d.f. Erlang(n) with the same scale parameter β, and are independent of each other. If the first claim has occurred at time 0 and the ruin has not occurred, the risk model is also a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate.

We denote by $\phi_{k;\delta,\alpha}$, $k \le n$, the penalty function corresponding to the delayed renewal risk model, where the case of $k = n$ is just $\Phi_{n;\delta,\alpha}(u)$: $\phi_{n;\delta,\alpha} \equiv \Phi_{n;\delta,\alpha}(u)$.

In [7] the following theorem about an integro-differential equation for the Gerber-Shiu function in a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate was proved.

Theorem 3.1. Consider a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate and the relative safety loading condition (2.2) holds. If

$$\int_0^\infty \int_0^\infty \omega(x_1, x_2) f(x_1 + x_2) dx_1\, dx_2 < \infty,$$

then $\Phi_{n;\delta,\alpha}(u)$ satisfies the following integro-differential equation:

$$\beta^n \int_0^u \Phi_{n;\delta,\alpha}(u-x) f(x) dx + \beta^n \int_u^\infty \omega(u, x-u) f(x) dx =$$
$$= [(\alpha+\beta)I - (\delta u + c)D]^n \Phi_{n;\delta,\alpha}(u), \qquad (3.1)$$

where I and D denote the identity operator and differentiation operator, respectively, and

$$[(\alpha+\beta)I - (\delta u + c)D]^n \Phi_{n;\delta,\alpha}(u) =$$
$$= \sum_{m=0}^{n} C_n^m (\alpha+\beta)^{n-m} (-1)^m \sum_{l=0}^{m} V_m^l \delta^{m-l} \sum_{r=0}^{l} C_l^r \delta^r c^{l-r} u^r\, \Phi_{n;\delta,\alpha}^{(l)}(u),$$
$$C_n^m = \frac{n!}{m!\,(n-m)!},$$
$$V_m^0 = 0, \quad m \ge 1, \quad V_0^0 = 1,$$
$$V_m^1 = V_m^m = 1, \quad m \ge 1,$$
$$V_m^l = l V_m^{l-1} + V_{m-1}^{l-1}, \quad 2 \le l \le m-1, \quad m \ge 3,$$
$$\Phi_{n;\delta,\alpha}^{(l)}(u) = \frac{d^l}{du^l} \Phi_{n;\delta,\alpha}(u), \quad l \ge 1, \quad \Phi_{n;\delta,\alpha}^{(0)}(u) = \Phi_{n;\delta,\alpha}(u).$$

There was also obtained the following recurrent relation during the proof of the Theorem 3.1:

$$(\delta u + c)\phi'_{k;\delta,\alpha}(u) = -\beta \phi_{k-1;\delta,\alpha}(u) + (\alpha+\beta)\phi_{k;\delta,\alpha}(u), \quad 2 \le k \le n. \qquad (3.2)$$

Note that, if we take $\alpha = 0$ and $\omega(x_1, x_2) \equiv 1$, $\Phi_{n;\delta,\alpha}(u)$ reduces to the ruin probability $\Psi_{n;\delta}(u)$. Therefore, from Theorem 3.1 can be obtained the following

Corollory 3.1. Let the conditions of Theorem 3.1 be satisfied. Additionally, let $\alpha = 0$ and $\omega(x_1, x_2) \equiv 1$. Then the probability $\Psi_{n;\delta}(u)$ satisfies the following integro-differential equation:

$$\beta^n \int_0^u \Psi_{n;\delta}(u-x)f(x)dx + \beta^n \bar{F}(u) = [\beta I - (\delta u + c)D]^n \Psi_{n;\delta}(u).$$

Using the integro-differential equation (3.1), we can obtain a n-th order differential equation satisfied by Laplace transform of $\Phi_{n;\delta,\alpha}(u)$. Now define Laplace transform of a function $h(u)$ by

$$\Lambda\{h(u)\}(s) = \hat{h}(s) = \int_0^\infty e^{-su}h(u)du.$$

The main result of present study can be formulated in the following form.

Theorem 3.2. Consider a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate and the relative safety loading condition (2.2) holds. If $\omega(x_1, x_2)$ is bounded, then $\hat{\Phi}_{n;\delta,\alpha}(s)$ satisfies the following n-th order differential equation:

$$\sum_{i=0}^{n} \hat{\Phi}^{(i)}_{n;\delta,\alpha}(s) K_{n,i}(s) - \beta^n \hat{f}(s)\hat{\Phi}_{n;\delta,\alpha}(s) = \sum_{i=0}^{n-1} \Phi^{(i)}_{n;\delta,\alpha}(0) Q_{n,i}(s) + \beta^n \hat{\omega}(s), \qquad (3.3)$$

where $s > 0$ and

$$\hat{\Phi}^{(i)}_{n;\delta,\alpha}(s) = \frac{d^i}{ds^i}\hat{\Phi}_{n;\delta,\alpha}(s), \qquad i \geq 1, \qquad \hat{\Phi}^{(0)}_{n;\delta,\alpha}(s) \equiv \hat{\Phi}_{n;\delta,\alpha}(s)$$

$$\Phi^{(i)}_{n;\delta,\alpha}(0) = \frac{d^i}{du^i}\Phi_{n;\delta,\alpha}(u)\bigg|_{u=0}, \qquad i \geq 1, \qquad \Phi^{(0)}_{n;\delta,\alpha}(0) = \Phi_{n;\delta,\alpha}(0),$$

$$K_{n,i}(s) = \sum_{m=i}^{n} C_n^m (\alpha+\beta)^{n-m}(-1)^m \sum_{l=i}^{m} V_m^l \delta^{m-l} \sum_{r=i}^{l} (-1)^r C_l^r C_r^i A_l^{r-i} \delta^r c^{l-r} s^{l-r+i},$$
$$0 \leq i \leq n,$$

$$Q_{n,i}(s) = \sum_{m=i+1}^{n} C_n^m (\alpha+\beta)^{n-m}(-1)^m \sum_{l=i+1}^{m} V_m^l \delta^{m-l} \sum_{r=0}^{l-1-i} (-1)^r C_l^r A_{l-1-i}^r \delta^r c^{l-r} s^{l-1-i-r},$$
$$0 \leq i \leq n-1,$$

$$A_l^r = \frac{l!}{(l-r)!},$$

$$\hat{\omega}(s) = \Lambda\{\omega(u)\}(s),$$

$$\omega(u) = \int_u^\infty \omega(u, x-u)f(x)dx.$$

Proof. Since $\omega(x_1, x_2)$ is bounded, that is there exists a constant $M > 0$ such that $\omega(x_1, x_2) \leq M$. Then $\phi_{k;\delta,\alpha}(u) \leq M$ for all $u \geq 0$ and for every $k \leq n$, which implies that $\hat{\phi}_{k;\delta,\alpha}(u) < \infty$ for all $s > 0$ and for every $k \leq n$. It is clear that for every $k \leq n$

$$0 \leq \lim_{u\to\infty} \phi_{k;\delta,\alpha}(u) \leq M \lim_{u\to\infty} \psi_{k;\delta}(u),$$

where the last inequality is implied by the relative safety loading condition. Thus for every $k \leq n$

$$\lim_{u\to\infty} \phi_{k;\delta,\alpha}(u) = 0.$$

Furthermore, it follows recursively from (3.2) that for $l = 1,2, \dots, n$

$$\lim_{u\to\infty} \Phi_{n;\delta,\alpha}^{(l)}(u) = 0.$$

So, Laplace transform of a function $u^r\Phi_{n;\delta,\alpha}^{(l)}(u)$, $r \le l \le n$, exists and

$$\Lambda\left\{u^r\Phi_{n;\delta,\alpha}^{(l)}(u)\right\}(s) = \\ = (-1)^r\left[\sum_{i=0}^{r} C_r^i A_l^{r-i} s^{l-r+i}\widehat{\Phi}_{n;\delta,\alpha}^{(i)}(s) - \sum_{i=0}^{l-1-r} A_{l-1-i}^{r} s^{l-1-i-r}\Phi_{n;\delta,\alpha}^{(i)}(0)\right]. \qquad (3.4)$$

It is clear that the conditions of Theorem 3.1 hold. So (2.1) satisfies (3.1). Taking Laplace transform on both sides of equation (3.1), and taking into consideration (3.4), then changing order of summations yields us to (3.3).

This completes the proof of Theorem 3.2.

Remark 3.1. Taking $\alpha = 0$ and $\omega(x_1, x_2) \equiv 1$ in (3.1) we can obtain the differential equation for the Laplace transform of the ruin probability $\Psi_{n;\delta}(u)$, denoted by $\widehat{\Psi}_{n;\delta}(s)$.

Remark 3.2. Taking $n = 2$, from (3.1) we can obtain the result in Gaoqin, et al [6].

4. Conclusion. In this paper the expected discounted penalty function of a Sparre Andersen risk model with Erlang(n)-distributed interclaim times and a constant interest rate is investigated and a n-th order differential equation which the Laplace transform of the Gerber-Shiu function satisfies is derived. For this an integro-differential equation for the Gerber-Shiu function is used. Finally, it is shown that how we can obtain previously known results in the literature.

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UOT 518.2

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Sabit faiz dərəcəli Sparre Andersen risk modelində Gerber-Shiu funksiyasının Laplas çevirməsi haqqında

Ardıcıl sığorta hadisələri arasındakı zamanı Erlanq(n) qanunu ilə paylanan sabit faiz dərəcəli Sparre Andersen sığorta riski modeli araşdırılmış, Gerber-Shiu funksiyasının Laplas çevirməsi üçün diferensial tənlik qurulmuşdur.

Açar sözlər: Gerber-Shiu funksiyası, Sparre Andersen risk modeli, Erlanq(n) risk prosesi, Laplas çevirməsi, faiz dərəcəsi, inteqro-diferensial tənlik, n tərtibli diferensial tənlik

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О преобразовании Лапласа функции Гербер-Шью в модели риска Спарре Андерсена с постоянной процентной ставкой

Исследована модель страхования риска Спарре Андерсена со временем между требований распределенными по закону Эрланга(n) и с постоянной процентной ставкой. Построено дифференциальное уравнение для преобразования Лапласа функции Гербер-Шью.

Ключевые слова: функция Гербер-Шью, модель риска Спарре Андерсена, модель риска Эрланга(n), преобразование Лапласа**,** процентная ставка, интегро-дифференциальное уравнение, дифференциальное уравнение n-го порядка

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