Informatics and Control Problems 39 Issue 1 (2019) 3-15
journal homepage: www.icp.az

On the application of the hybrid methods to the solving of the Volterra integro-differential equations

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ARTICLE INFO

Article history:
Received 13.05.2019
Received in revised form 30.05.2019
Accepted 10.06.2019
Available online 12.06.2019

Keywords:
Volterra integro-differential equations
Hybrid method
Degree and stability
Necessary and sufficient conditions
Initial-value problem

ABSTRACT

As is known, in the early 20th century, for solving of some problems in the field of natural sciences, Vito Volterra faced the necessity for solving integro-differential equations with variable boundaries. Several authors studied integro-differential equations in junction of differential and integral equations. Therefore, they used quadrature method or its modification. Among these, stable methods that have a high order of accuracy was of theoretical and practical interest presented. In this paper, hybrid methods are proposed for the construction of numerical methods with this property. The authors construct concrete second derivative multistep methods with the orders of accuracy of $p = 6$ and $p = 8$ using information pertaining to the solution of the considered problem with one and two mesh points, respectively. Also here, any algorithm can be formulated for using the constructed methods. A comparison of the hybrid methods with the Gauss method is given.

1. Introduction

In the 1930s, Volterra showed that mathematical models for some seasonal diseases, e.g., influenza, are formulated as integral and differential equations (see [1, pp.22-34]); this work gave an impetus to the development of approximate methods for solving integro-differential equations. One popular method for solving what are now known as Volterra integro-differential equations is the method of quadratures. Note that the quadrature method was first used by Volterra to solve integro-differential equations with variable boundaries.

Consider the following initial-value problem in Volterra integro-differential equations:

$$y' = F(x, y, z(x)),\ y(x_0) = y_0,\ x_0 \le x \le X, \tag{1}$$

where $y(x)$ is a solution of the problem. The function $z(x)$ is defined as follows:

$$z(x) = \int_{x_0}^{x} K(x, s, y(s))ds,\ \ x_0 \le s \le x \le X. \tag{2}$$

Obviously, if the function $z(x)$ is known, then problem (1) can be rewritten as:

$$y' = f(x, y(x)),\ y(x_0) = y_0. \tag{3}$$

Therefore, using the known quantities $y_1, y_2, \dots y_{k-1}$ and $z_1, z_2, \dots z_{k-1}$ to solve problem (1)

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www.icp.az/2019/1-01.pdf

can be done by applying the k-step method with constant coefficients. Then, we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i F(x_{n+i}, y_{n+i}, z_{n+i}), \tag{4}$$

Here, $y_m, z_m (m = 0,1,2, \dots)$ are the approximate values of the function $y(x)$ and $z(x)$ on the mesh points $x_m = x_0 + mh, (m = 0,1,2, \dots)$, where the parameter $h > 0$ is the integration step size, which is divided the segment $[x_0, X]$ into N equal parts. It is easy to see that if there is a way to determine $z_{n+k} (n \geq 0)$, then it should be used in formula (4). Furthermore, we can calculate the values of the function $y(x)$ on the mesh points $x_{n+k} (n = 0,1,2, \dots, N - k)$. In this case, solving problem (1) is equivalent to solving an initial-value problem for ordinary differential and integral equations (see, e.g., [2-6]). As is known, using the Volterra integro-differential equations, one can study the memory of the land. By solving some seismology problems also one can study the memory of the land. For instance, using the information about oil and gas producing facilities which have been fundamentally investigated in [7-9]. Thus, by means of multistep methods with constant coefficients, we can find the solution of the problem (1). Note that solving integral equations can be accomplished with several different approximation methods (see, e.g., [10-12]). In the class of problems (1), the most basic research is on the following problem:

$$y' = f(x, y) + \int_{x_0}^{x} K(x, s, y(s))ds, \;\; y(x_0) = y_0, \;\; x_0 \leq s \leq x \leq X. \tag{5}$$

To solve problem (5), one can use the following multistep method (see, e.g., [13] or [14]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\gamma_i z_{n+i} \;, \tag{6}$$

$$\sum_{i=0}^{k}\hat{\alpha}_i z_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \tag{7}$$

This method is obtained by using a multistep method to solve both integral equation (2) and initial-value problem (5). Therefore, solving problem (1) can be accomplished with one of the approximate methods of ordinary differential equations by employing some combinations of the methods proposed for solving integral equations with variable boundaries. The order of accuracy of the stable methods, which is constructed by scheme (6)-(7), does not exceed $k + 2$; this result was established by Dahlquiste (see [15]). Therefore, scientists have proposed various ways to construct stable methods with an order of accuracy greater than $k + 2$. To this end, in [16] a hybrid method first investigated by Gear and Butcher was applied to solve problem (1) (see [17, 18]). However, in [14], the existence of stable forward jumping methods with an order of accuracy higher than $k + 2$ was proved and a method to solve Volterra integral equations was proposed. It should be noted that in [19], stable hybrid methods with an order of accuracy higher than $2k$ were constructed, but in [20], a hybrid method was applied to extend Makroglou's ideas for solving equation (2). Thus, we find that the numerical methods of ordinary differential equations can be applied to solve both integral equations of type (2) and initial-value problems of form (1). Note that if one wishes to solve Volterra integral equations using quadrature or other methods that are different from method (7), one cannot exclusively use the methods of ordinary differential equations to solve problem (1). However, if the kernel of the integral is a degenerate function, i.e., if

$$K(x, z, y) = \sum_{\nu=1}^{m} a_\nu(x) b_\nu(z, y), \tag{8}$$

then problem (1) can be reduced to a system of ordinary differential equations. Obviously, in this case problem (1) can be solved using the methods of ordinary differential equations.

In this paper, we construct stable hybrid methods with a high order of accuracy that used information about the solution of problem (1) only minimally. The proposed work is a continuation of the investigations conducted in [19].

Consider the application of the following method to solving problem (1):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\left(\beta_i y'_{n+i} + \gamma_i y'_{n+i+\nu_i}\right), \ \left(|\nu_i| < 1; \ i = 0,1,\dots,k\right), \tag{9}$$

from which we may obtain many well-known hybrid methods. In [21], method (9) is applied to solve problem (3), and it was proved that there are stable methods of type (9) with degree $p = 3k + 1$. For the construction with the more exact methods, using hybrid methods with the second derivative is considerd.

2. Problem statement

One of the basic problems in modern computational mathematics is the construction of stable methods with higher accuracy. Therefore, here by the comparison of the known numerical methods we show the advantages and disadvantages of these methods. For removing from the indicated disadvantages, we construct here stable multistep methods, which are applied to solve the initial-value problem of the Volterra integro-differential equations. For this purpose, we show that the accuracy of the known multistep method is bounded by the value of $k + 2$. And to construct more accurate methods, it is proposed to use the Hybrid methods and prove the existence of stable methods of the Hybrid type with the degree $p = 3k + 3$. By indicating some disadvantages of these methods, it is proposed to use the multistep second derivative methods with constant coefficients. We also consider constructing of multistep second derivative methods of the hybrid type. Here we construct the stable methods with the degree $p = 10$ for the $k = 1$ which are applied to the solving of the model problem.

3. Methods of solution

Among the numerical methods converging methods of are both theoretical and practical interest. It is known that stability is a necessary and sufficient condition for the convergence of multistep methods. Thus, we investigate the stable hybrid methods that are used to solve problem (1). Usually, a study of multistep methods imposes certain restrictions on the coefficients (see, e.g., [15]). These constraints on the coefficients of method (9) can be written in the following form:

A. The values of the variables $\alpha_i, \beta_i, \gamma_i, \nu_i$ $(i = 0,1,\dots,k)$ are real numbers and $\alpha_k \neq 0$.

B. The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i\lambda^i; \ \ \vartheta(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i; \ \ \gamma(\lambda) \equiv \sum_{i=0}^{k}\gamma_i\lambda^{i+\nu_i}$$

of method (9) have no common multiplier that is not a constant.

C. $\vartheta(1) + \gamma(1) \neq 0$ and $p \geq 1$.

Here, p is the order of accuracy of method (9), which is defined in the following form:

Definition 1. For a sufficiently smooth function $y(x)$, method (9) has the degree $p > 0$ if the following holds:

$$\sum_{i=0}^{k}\alpha_i y(x+ih) - h\sum_{i=0}^{k}\left(\beta_i y'(x+ih) + \gamma_i y'(x+(i+\nu_i)h)\right) = O(h^{p+1}), \ h \to 0. \tag{10}$$

Condition A is obvious. Therefore, we consider condition B and assume the reverse. Then, the polynomials $\rho(\lambda), \vartheta(\lambda)$, and $\gamma(\lambda)$ have a common multiplier, which we denote by $\varphi(\lambda)$. After taking into account the shift operator $E \cdot (E^{\nu}y(x) = y(x + \nu h))$, the finite-difference equation (9) can be rewritten as follows:

$$\rho(E)y_n - h\vartheta(E)y'_n - h\gamma(E)y'_n = 0. \tag{11}$$

Let us use the given assumptions to rewrite equation (11) in the following form:

$$\varphi(E)(\rho_1(E)y_n - h\vartheta_1(E)y'_n - h\gamma_1(E)y'_n) = 0.$$

Here,

$$\rho_1(\lambda) = \rho(\lambda)/\varphi(\lambda); \;\; \vartheta_1(\lambda) = \vartheta(\lambda)/\varphi(\lambda); \;\; \gamma_1(\lambda) = \gamma(\lambda)/\varphi(\lambda).$$

Hence, we find that

$$\rho_1(E)y_n - h\vartheta_1(E)y'_n - h\gamma_1(E)y'_n = 0, \tag{12}$$

because $\varphi(\lambda) \neq const$. Obviously, to have a unique solution to finite-difference equation (12), there should be no more than $k-1$ initial data. However, from the theory of finite-difference equations it is known that for a finite-difference equation of order k to have a unique solution, k initial data are required. However, difference equations (12) and (9) are equivalent. Hence, difference equation (9) has a unique solution despite having no more than $k-1$ initial data, which contradicts the above-mentioned theory. Consequently, the assumption that there is a common factor of the polynomials $\rho(\lambda), \vartheta(\lambda)$, and $\gamma(\lambda)$ is incorrect. Now, consider the validity of condition C. Assume that method (9) converges. Then, as (9) approaches the limit and as $h \to 0$ we have:

$$\sum_{i=0}^{k} \alpha_i y(x) = 0, \; (x = x_0 + nh). \tag{13}$$

Because $y(x) \neq 0$, from equation (13) we have:

$$\rho(1) = 0\,. \tag{14}$$

Equation (14) is a necessary condition for the convergence of the method defined by formula (9), and by using it we can write

$$\rho(\lambda) = (\lambda - 1)\rho_1(\lambda)\,.$$

Furthermore, by using (11) we obtain:

$$\rho_1(E)(y_{j+1} - y_j) - h\vartheta(E)y'_j - h\gamma(E)y'_j = 0. \tag{15}$$

Here, by changing the value of variable j from 0 to n and summing the resulting equations, we obtain:

$$\rho_1(E)(y_{n+1} - y_0) - h\vartheta(E)\sum_{j=0}^{n} y'_j - h\gamma(E)\sum_{j=0}^{n} y'_j = 0.$$

Then, as $h \to 0$, we have:

$$\rho_1(1)(y(x) - y_0) = (\vartheta(1) + \gamma(1))\int_{x_0}^{x} y'(\xi)d\xi. \tag{16}$$

However, from problem (1) we can write:

$$y(x) = y(x_0) + \int_{x_0}^{x} f(\xi, y(\xi))d\xi. \tag{17}$$

By comparing (16) and (17), it is clear that

$$\rho_1(1) = \rho'(1) = \vartheta(1) + \gamma(1).$$

It is easy to prove that due to the conditions

$$\rho(1) = 0; \;\; \rho'(1) = \vartheta(1) + \gamma(1),$$

consequence is that $p \geq 1$.

Now we must prove that $\vartheta(1) + \gamma(1) \neq 0$. Assume otherwise. Then, from the conditions $\rho(1) = 0$ and $\rho'(1) = 0$ we obtain that $\lambda = 1$ is a double root of the polynomial $\rho(\lambda)$.

Consider the homogeneous finite-difference equation

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + ... + \alpha_1 y_{n+1} + \alpha_0 y_n = 0,$$

whose general solution can be written in the following form:

$$y_m = c_1\lambda_1^m + c_1 m\lambda_1^m + c_3\lambda_3^m + \ldots + c_k\lambda_k^m,$$

where λ_i $(i = 1,2,\ldots,k)$ are the roots of the polynomial $\rho(\lambda)$. Hence, as $h \to 0$ we know that $y_m \to \infty$ because $m \to \infty$. Thus, if $\vartheta(1) + \gamma(1) = 0$, then the method does not converge. It follows that $\vartheta(1) + \gamma(1) \neq 0$. If we use the conditions

$$\rho(1) = 0 \text{ and } \vartheta(1) + \gamma(1) = \rho'(1)$$

in asymptotic relation (10), then we obtain that

$$p \geq 1.$$

Now, consider using method (9) to solve problem (1). To this end, we investigate the numerical solution of problem (1) by using the following methods:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i F_{n+i} + h\sum_{i=0}^{k} \gamma_i F_{n+i+\nu_i}\ , \tag{18}$$

$$\sum_{i=0}^{k} \alpha_i z_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) + h\sum_{j=0}^{k}\sum_{i=0}^{k} \gamma_i^{(j)} K(x_{n+j+m_j}, x_{n+i+\nu_i}, y_{n+i+\nu_i}),$$

$$(|m_j| < 1,\ j = 0,1,\ldots,k). \tag{19}$$

To study method (19), we suggest that the kernel of the integral $K(x,s,y)$ is a continuous function that is defined in the region $G = \{x_0 \leq s \leq x \leq X, |y| \leq b\}$ and that has continuous derivatives up to and including some order p. If in method (19) we take into account the properties of the function $K(x,s,y)$, then we have following (see, e.g., [20]):

$$\beta_i^{(j)} = 0;\ \gamma_i^{(j)} = 0\ \ (i > j).$$

Method (19) as a numerical method for solving Volterra integral equations is studied in [20]. We remark that method (18) is a generalization of hybrid methods. In the past few years, scientists have thoroughly studied the application of hybrid methods to solving initial-value problems with ordinary differential equations and Volterra integro-differential equations (see, e.g., [16-24]). Let us consider finding the coefficients in methods (9) and (19).

It can be shown that by using the Taylor expansions

$$y(x+ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2!}y''(x) + \ldots + \frac{(ih)^p}{p!}y^{(p)}(x) + O(h^{p+1}), \tag{20}$$

$$y'(x+l_ih) = y'(x) + l_ihy''(x) + \frac{(l_ih)^2}{2!}y'''(x) + \ldots + \frac{(l_ih)^{p-1}}{(p-1)!}y^{(p)}(x) + O(h^p), \tag{21}$$

in asymptotic equation (10), we can obtain the necessary and sufficient conditions for equation (10), where $x = x_0 + nh$ is the fixed point and $l_i = i + \nu_i$ $(i = 0,1,2,\ldots,k)$. These conditions can be written in the form of systems of equations that consist of the following nonlinear equations:

$$\sum_{i=0}^{k} \alpha_i = 0,\ \sum_{i=0}^{k} (i\alpha_i - \beta_i - \gamma_i) = 0,$$

$$\sum_{i=0}^{k} \left(\frac{i^l}{l!}\alpha_i - \frac{i^{l-1}}{(l-1)!}\beta_i - \frac{(i+\nu_i)^{l-1}}{(l-1)!}\gamma_i\right) = 0,\ \ (l = 2,3,.,\ldots,p). \tag{22}$$

It is easy to determine that system (22) for the values $\nu_i = 0$ $(i = 0,1,\ldots,k)$ is linear and coincides with known systems that are used to determine the coefficients of the multistep method with constant coefficients. Furthermore, for the conditions $|\nu_0| + |\nu_1| + \cdots + |\nu_k| \neq 0$, system (22) is nonlinear; by solving it, we determine the coefficients of method (9). In this system, the number of unknowns is equal to $4k + 4$ and the number of equations is equal to $p + 1$. Because system (22) is homogeneous, it always has a trivial solution, but to ensure that system (22) will have a solution that is different from zero, the condition $4k + 4 > p + 1$ must hold. Thus, the following can be written:

$$p \le 4k + 2.$$

Note that if we take $\beta_i = 0$ $(i = 0,1,2,\dots,k)$, then the relationship between the degree and the order of method (9) will be as follows:

$$p \le 3k + 1 .$$

It is known that if we consider the case $\gamma_i = 0$ $(i = 0,1,2,\dots,k)$, then the degree of the stable method received from formula (9) satisfies the condition $p \le 2[k/2] + 2$ (see [15]).

To determine the coefficients in method (19), consider a special case and let $K(x,s,y) = F(x,y)$. Then, from (2) we have

$$\vartheta' = F(x,y), \quad \vartheta(x_0) = 0. \tag{23}$$

If we apply method (19) to solve problem (23), then we obtain:

$$\sum_{i=0}^{k} \hat{\alpha}_i \vartheta_{n+i} = h\sum_{i=0}^{k} \hat{\beta}_i F_{n+i} + h\sum_{i=0}^{k} \hat{\gamma}_i F_{n+i+\nu_i} , \tag{24}$$

where

$$\sum_{j=0}^{k} \beta_i^{(j)} = \hat{\beta}_i; \quad \sum_{j=0}^{k} \gamma_i^{(j)} = \hat{\gamma}_i \quad (i = 0,1,\dots,k). \tag{25}$$

First, from system (22) we determine the values of $\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i, \nu_i$ $(i = 0,1,2,\dots,k)$, and then, by solving system (25) we find the coefficients of method (19). Note that in system (25), the number of equations is equal to $k+1$ and the number of unknowns is greater than $k+1$. Consequently, the solution of system (25) is not unique. Therefore, although the method of type (9) may be unique, the corresponding method of type (19) is not unique. This fact allows us to select some of the coefficients to construct the method with an extended region of stability.

Consider special cases and let $k = 1$. Then, by solving system (22) and using the solution in system (25), we obtain a few methods of degree $p = 6$. One of them is the following:

$$y_{n+1} = y_n + h(F_{n+1} + F_n)/12 + 5h(F_{n+1/2-\alpha} + F_{n+1/2+\alpha})/12, \ (\alpha = \sqrt{5}/10) \tag{26}$$

where the corresponding method of type (19) in one variant can be written as:

$$\begin{gathered} z_{n+1} = z_n + h(2K(x_{n+1}, x_{n+1}, y_{n+1}) + K(x_{n+1}, x_n, y_n) + K(x_n, x_n, y_n))/24 + +5h(K(x_{n+1}, x_{n+1/2+\alpha}, y_{n+1/2+\alpha}) + \\ + K(x_{n+1/2+\alpha}, x_{n+1/2+\alpha}, y_{n+1/2+\alpha}) + +K(x_{n+1}, x_{n+1/2-\alpha}, y_{n+1/2-\alpha}) + +K(x_{n+1/2-\alpha}, x_{n+1/2-\alpha}, y_{n+1/2-\alpha}))/24, \ (\alpha = \sqrt{5}/10). \end{gathered} \tag{27}$$

Note that the method with degree $p = 8$ for $k = 2$ is as follows:

$$y_{n+2} = y_n + h(64y'_{n+2} + 98y'_{n+1} + 18y'_n)/180 + h(18y'_{n+1+\beta} + 98y'_n + 64y'_{n+1-\beta})/180, \tag{28}$$

where the value $\beta = \sqrt{21}/14$.

Here we consider the study of some multistep methods by using results of the work [25]. To this end, we use some of the generalization of the method (2) which can be written as follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} (\beta_i F(x_{n+i}, y_{n+i}, z_{n+i}) + \gamma_i F(x_{n+\nu_i}, y_{n+\nu_i}, z_{n+\nu_i})). \tag{29}$$

This method in some sense coincides with the method (9). Because it can be assumed that there are stable methods of type (29) having a degree $p = 3k + 1$. However, stable methods of type (4) have a degree $p \le k + 2$. A simple comparison shows that the hybrid methods are more promising. Note that for the construction of more exact methods, some scholars suggest to use multistep methods with a second derivative (see eg. [26-29]). We know that stable methods of this type have a degree $p \le 2k + 2$. Consequently, such this modification of the multistep method does not extend the area of its applications. Therefore, here we propose constructing multistep methods at the junction of the hybrid method with the method of the second derivative, which can be written as follows:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}(\beta_i F_{n+i} + \hat{\beta}_i F_{n+i+\nu_i}) + h^2\sum_{i=0}^{k}(\gamma_i g_{n+i} + \hat{\gamma}_i g_{n+i+\nu_i}). \quad (30)$$

Here the coefficients $\alpha_i, \beta_i, \widehat{\beta_i}, \gamma_i, \widehat{\gamma_i}, \nu_i$ $(i = 0,1,2,\dots,k)$ are real numbers and $\alpha_k \neq 0$, but the function $g(x,y,z) = y''$ or $g(x,y,z) = K'_x(x,y,z) + K'_y(x,y,z)y' + K'_z(x,y,z)z'$. From the method (30) follows method (29), for the values $\gamma_i = \widehat{\gamma_i} = 0 \cdot (i = 0,1,2,\dots,k)$, but for the values $\widehat{\beta_i} = \widehat{\gamma_i} = 0 \cdot (i = 0,1,2,\dots,k)$ the method (30) follows second derivative multistep methods, which have investigated in some works (see, e.g. [26-29]). Note that stable second derivative multistep methods have order of accuracy $p \leq 2k + 2$. These methods in [26] are generalized as follows:

$$\sum_{i=0}^{m}\alpha_i y_{n+i} = h\sum_{i=0}^{l}\beta_i y'_{n+i} + h^2\sum_{i=0}^{k}\gamma_i y''_{n+i} \quad (\alpha_m \neq 0). \quad (31)$$

Depending on the relation between of the variables m, l and k different methods can be obtained from formula (31). For instance, for $m = l = k$ and $|\beta_k| + |\gamma_k| \neq 0$ implicit methods can be obtained from method (31), but $m = l = k$ and $\beta_k = \gamma_k = 0$ explicit methods can be obtained from method (31). It is clear that some connections can be established between methods (30) and (31). However, our aim is to study method (30). Typically, the study of a numerical method consists in determining the area of its application, finding the maximum value of its order of accuracy, determining the boundary of step size h or determining the stability region of the investigated method. Methods with the basic characteristic of the solution of considered problems are constructed. All of these listed properties of the numerical methods are determined by the value of the coefficient of method (30). Therefore, we consider determining the values of coefficients in method (30). After the application of the method of undetermined coefficients in order to find the values of the coefficients in method (30), we obtain a system of nonlinear algebraic equations, which generalizes the existing similar systems, such as system (22). To construct a system of algebraic equations, method (30) can be rewritten as follows:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}(\beta_i y'_{n+i} + \hat{\beta}_i y'_{n+i+\nu_i}) + h^2\sum_{i=0}^{k}(\gamma_i y''_{n+i} + \hat{\gamma}_i y''_{n+i+\nu_i}). \quad (32)$$

Using the diagram in Section 2 to find the values of coefficients in method (32), we obtain the following system of algebraic equations:

$$\sum_{i=0}^{k}\alpha_i = 0; \quad \sum_{i=0}^{k}(\beta_i + \beta'_i) = \sum_{i=0}^{k} i\alpha_i,$$

$$\sum_{i=0}^{k}\left(\frac{i^{l-1}}{(l-1)!}\beta_i + \frac{(i+\nu_i)^{l-1}}{(l-1)!}\hat{\beta}_i\right) + + \sum_{i=0}^{k}\left(\frac{i^{l-2}}{(l-2)!}\gamma_i + \frac{(i+\nu_i)^{l-2}}{(l-2)!}\hat{\gamma}_i\right) = \sum_{i=0}^{k}\frac{i^l}{l!}\alpha_i, \quad (33)$$

$$(l = 2,3,..,p).$$

The number of equations in system (33) is $p + 1$, and the number of unknowns is equal to $6k + 6$. Solving this system of equations, we determine the values of coefficients in method (32). It is obvious that the system always has zero solutions, which is of no interest to us. Therefore, it can be shown that there are methods with a degree $p = 6k + 4$. Methods of construction of type (32) are stable for $k = 1$, but for the $k = 2$, stable methods can be obtained from method (32) for $\alpha_2 = -\alpha_1 = 1$; $\alpha_0 = 0$ and $\alpha_2 = -\alpha_0 = 1$; $\alpha_1 = 0$. There are other cases when the methods obtained from (32) will be stable. For instance, if $\alpha_2 = 1, \alpha_1 = \alpha_0 = -1/2$. We are here to construct some specific methods having a degree $p \leq 10$ and compare these methods with well-known ones, as well as to investigate types of these methods (that is, they belong to a class of explicit or implicit methods). Now consider the case $k = 1$. Then from system (33) we obtain the following:

$$\beta_1 + \beta_0 + \hat{\beta}_1 + \hat{\beta}_0 = 1,$$

$$
\begin{aligned}
&\gamma_1+\gamma_0+\hat{\gamma}_1+\hat{\gamma}_0+\beta_1+l_1\hat{\beta}_1+l_0\hat{\beta}_0=\frac{1}{2};\\
&2(\gamma_1+l_1\hat{\gamma}_1+l_0\hat{\gamma}_0)+\beta_1+l_1^2\hat{\beta}_1+l_0^2\hat{\beta}_0=\frac{1}{3};\\
&3(\gamma_1+l_1^2\hat{\gamma}_1+l_0^2\hat{\gamma}_0)+\beta_1+l_1^3\hat{\beta}_1+l_0^3\hat{\beta}_0=\frac{1}{4};\\
&4(\gamma_1+l_1^3\hat{\gamma}_1+l_0^3\hat{\gamma}_0)+\beta_1+l_1^4\hat{\beta}_1+l_0^4\hat{\beta}_0=\frac{1}{5};\\
&5(\gamma_1+l_1^4\hat{\gamma}_1+l_0^4\hat{\gamma}_0)+\beta_1+l_1^5\hat{\beta}_1+l_0^5\hat{\beta}_0=\frac{1}{6};\\
&6(\gamma_1+l_1^5\hat{\gamma}_1+l_0^5\hat{\gamma}_0)+\beta_1+l_1^6\hat{\beta}_1+l_0^6\hat{\beta}_0=\frac{1}{7};\\
&7(\gamma_1+l_1^6\hat{\gamma}_1+l_0^6\hat{\gamma}_0)+\beta_1+l_1^7\hat{\beta}_1+l_0^7\hat{\beta}_0=\frac{1}{8};\\
&8(\gamma_1+l_1^7\hat{\gamma}_1+l_0^7\hat{\gamma}_0)+\beta_1+l_1^8\hat{\beta}_1+l_0^8\hat{\beta}_0=\frac{1}{9};\\
&9(\gamma_1+l_1^8\hat{\gamma}_1+l_0^8\hat{\gamma}_0)+\beta_1+l_1^9\hat{\beta}_1+l_0^9\hat{\beta}_0=\frac{1}{10}.
\end{aligned}
\tag{34}
$$

Solving the resulting system for $l_i = i + \nu_i\ (i = 0,1,\dots,k)$, we can construct a type of methods (32) with the degree $p \le 10$. First construct methods of type (32), do not put any restrictions on their coefficients. It is clear that without loss of generality we can take it as $\alpha_k = 1$. Then from (32) we have:

$$
y_{n+1}=y_n+h(\beta_0y'_n+\beta_1y'_{n+1}+\hat{\beta}_0y'_{n+\nu_0}+\hat{\beta}_1y'_{n+1+\nu_1})+h^2(\gamma_0y''_n+\gamma_1y''_{n+1}+\hat{\gamma}_0y''_{n+\nu_0}+\hat{\gamma}_1y''_{n+1+\nu_1}). \tag{35}
$$

Note that for $\alpha_1 = 1$ from the necessary condition $\rho(1) = 0$ of the convergence of method (32) we determine that $\alpha_0 = -1$. By means of the selection coefficient in (35), we obtain different methods. Take into account that the complexity of computing the values of variables $y''(x)$ in some cases any of the coefficients of method (35) may be ignored. Therefore, these coefficients here are equal to zero. To this end, consider the following cases:

Variant I. $(p = 10)$.

$$
\begin{aligned}
&\beta_1 = 0.122153479207037 44;\ \beta_0 = 0.15666106\ 5399144\ ;\\
&\hat{\gamma}_1 = 0.000948172618515 75;\ \hat{\gamma}_0 = \text{-}0.0000071\ 9154282459\ ;\\
&\gamma_1 = \text{-}0.004349463441297 16;\ \gamma_0 = 0.00748767\ 300880619\ ;\\
&\hat{\beta}_1 = 0.34229328\ 05307209\ ;\ \hat{\beta}_0 = 0.37889217\ 49062838\ ;\\
&l_1 = 0.71344424622241004;\ l_0 = 0.34194466\ 714378163\ ;
\end{aligned}
$$

Variant II. $(\gamma_1 = \widehat{\gamma_1} = \widehat{\gamma_0} = 0; p = 9)$.

$$
\begin{aligned}
&\beta_1 = 0.05819736304284103;\ \beta_0 = 0.19190172\ 45988311\ ;\\
&\gamma_0 = 0.01149880\ 442542037\ ;\ \hat{\beta}_1 = 0.32222913\ 09798016\ ;\\
&\hat{\beta}_0 = 0.42767178\ 13784907\ ;\ l_1 = 0.79729845579010 02;\\
&l_0 = 0.40543017\ 22504762
\end{aligned}
$$

Variant III. $(\widehat{\beta_1} = \widehat{\gamma_1} = 0; p = 10)$.

$$\beta_1 = 0.1812975097034355 3; \ \beta_0 = 0.29905999\ 40784864\ ;$$
$$\hat{\gamma}_0 = -0.00010121\ 84020778\ ; \gamma_1 = -0.0094756058148332 7;$$
$$\gamma_0 = 0.03028505\ 82307236\ ; \hat{\beta}_0 = 0.51964249\ 63166858\ ;$$
$$l_1 = 0.6449489742783179; \ l_0 = 0.57328482\ 89663591\ ;$$

Variant IV. $(\beta_1 = \gamma_1 = \widehat{\beta_1} = \widehat{\gamma_1} = 0; p = 4)$

$$\beta_0 = 0.42777172\ 72166551\ ; \ \hat{\gamma}_0 = -0.04393727\ 637262437\ ;$$
$$\gamma_0 = 0.06105439\ 761914382\ ; \ \hat{\beta}_0 = 0.57222827\ 27833449\ ;$$
$$l_1 = 0.6449489742783179; \ l_0 = 0.84386406\ 91497394\ ;$$

Each of these variant has some logical basis. For instance, for the application of methods obtained in the first variant to solve specific problems, it is the best option. It is easy to verify that the involvement quantity y_{n+a} $(|\alpha| < 1)$ in the multistep methods complicates their use. However, if the methods have unused quantities of the type y_{n+a} $(|\alpha| < 1)$, then for the orders of accuracy of these methods the following holds: $p \le 2k + 2$ (see e.g. [26-29]). Therefore, here we have identified the above-listed variants.

The use of the method (35) mainly depends on the construction of the formula for computing the value of the type y_{n+a} $(|\alpha| < 1)$ with a corresponding accuracy. It should be noted that an increase in the values of the order of accuracy of such intermediate formulas complicates its constructions. This brings up the issue of determining the optimal number of the mesh points used in these formulas. Usually, to calculate the values of variables y_{n+a} $(|\alpha| < 1)$, the construction of a more accurate formula is attempted, using a minimal amount of mesh points. In the basic formula with a high accuracy and using a minimal amount of mesh points is unstable. We here try to construct a concrete stable formula, which has high accuracy and uses a minimal amount of mesh points, i.e. to find the "golden mean" in the set of investigated formulas. Finding the the "golden mean" in a set of formulas is not always possible. This is why we aim to describe some ways that allow to construct formulas with the "golden mean". Sometimes, these formulas are required to expand the area of stability.

For simplicity, let us consider the application of the following method:

$$y_{n+1} = y_n + h(y'_{n+1/2+\alpha} + y'_{n+1/2-\alpha})/2, \quad (\alpha = \sqrt{3}/6). \tag{36}$$

This method will be used to solve problem (1); to use it, one must determine the values of $y_{n+1/2+\beta}, y_{n+1/2-\alpha}$, which can be achieved as follows:

$$y_{n+1/2+\beta} = y_{n+1/2} + ((4\beta^3 + 6\beta^2)\hat{y}'_{n+1} - (8\beta^3 - 24\beta)y'_{n+1/2} + (4\beta^3 - 6\beta^2)y'_n)/24. \tag{37}$$

Using the formula

$$\hat{y}_{n+1} = y_n + hy'_{n+1/2}, \tag{38}$$

we can find the variable $\hat{y}_{n+1}$. Then, we can determine $y_{n+3/2}$ from the following formula:

$$y_{n+3/2} = y_{n+1/2} + h(7y'_{n+1} - 2y'_{n+1/2} + y'_n)/6. \tag{39}$$

By using the following sequence of methods, one can solve initial-value problem (23).

Step 1. Calculate $\hat{y}_{n+1}$ from formula (38).

Step 2. Calculate $y_{n+1/2\pm\alpha}$ from formula (37).

Step 3. Calculate y_{n+1} from formula (36).

Step 4. Calculate $y_{n+3/2}$ from formula (39).

Now let us consider that the sequence consists of the some formulas, using which one can construct the algorithm for solving problem (1):

Step I.

$$y_{n+1+\gamma} = y_{n+1} + \gamma h y'_{n+1} + \frac{(\gamma h)^2}{360\alpha^3}((3\gamma^3 - 10\gamma\alpha^2)y''_{n+1+2\alpha} + (60\gamma\alpha^2 + 15\gamma^2\alpha - 9\gamma^3)y''_{n+1+\alpha} + (180\alpha^3 - 30\gamma\alpha^2 - \\ - 30\gamma^2\alpha + 9\gamma^3)y''_{n+1} + (-20\gamma\alpha^2 + 15\gamma^2\alpha - 3\gamma^3)y''_{n+1-\alpha}) \quad (40)$$

$$(R_n = \frac{\alpha\gamma^3 h^6}{1440}(20\alpha^3 - 5\alpha\gamma - 6\gamma^2)y^{VI}(\xi)),\ \xi \in (x_n, x_{n+1+2|\alpha|})$$

Step II.

$$y_{n+1+\gamma} = y_{n+1} + \gamma h y'_{n+1} + \frac{(\gamma h)^2}{12\alpha}(\gamma y''_{n+1+\alpha} + 6\alpha y''_{n+1} - \gamma y''_{n+1-\alpha}) \quad (R_n = O(h^5)) \quad (41)$$

If in the formula (40) put $\gamma = -\alpha$ and $\alpha = 1$, then receive the following:

$$y_{n+1-\alpha} = y_{n+1} - \alpha h y'_{n+1} + \frac{(\alpha h)^2}{360}(7y''_{n+1+2\alpha} - 36y''_{n+1+\alpha} + 171y''_{n+1} + 38y''_{n+1-\alpha}) \quad (42)$$

To calculate the values of the quantity y_{n+2} one can use the following:

Step III.

$$hy'_{n+1} = y_{n+1} - y_n + h^2(3y''_n - 2y''_{n+1} + \hat{y}''_{n+2})/24,\ (R_n = O(h^5))$$

Step IV.

$$\hat{y}_{n+2} = y_n + 2hy'_n + h^2(4y''_{n+1} + 2y''_n)/3, \quad (R_n = O(h^5))$$

Step V.

$$y_{n+2} = 2y_{n+1} - y_n + h^2(5y''_{n+1+\alpha} + 14y''_{n+1} + 5y''_{n+1-\alpha})/24,\ (p = 6,\ R_n = O(h^8),\ \alpha = \sqrt{10}/5) \quad (43)$$

It is shown that the constructed secondary formulas in the form of (40) are available. Because by choosing variable α one can change structures of the secondary formulas. For instance, for $\alpha = 1$ we obtain that to use formula (42), we must know the values y_{n+m} $(m = 0,1,2,3)$. But for the value $\alpha = 1\ /\ 2$ we must know the values $y_{n+m/2}$ $(m = 1,2,3,4)$. Consequently by choosing values of variable α, we can construct formulas with different characteristics.

To illustrate these results, consider the following table.

One may also consider and compare the results obtained by methods of type (9) with other known methods using the following model problems:

1. $y' = 1 + y - x\exp(-x^2) - 2\int_0^x xt\exp(-y^2(t))dt,\ 0 \le x \le 2,\ y(0) = 0$

(the exact solution is $y(x) = x$).

2. $y' = (4\exp(-y) - x^3)/3 + \frac{4}{3}\int_1^x \frac{1}{x}s^2\exp(y(s))ds,\ 1 \le x \le 2,\ y(1) = 0$

(the exact solution is $y(x) = ln\, x$).

Note that the obtained results are consistent with the theoretical results presented here, which can be found in Table 1.

Table 1
Results for the solution of problem 1

Number of example	x	By Gragg & Stetter (see[16])	By Kohfeld & Thomson (see[16])	By the method from [3]	For hybrid method (36)
$h = 0,05$	0.1 0.5 1.00	1.3E-09 5.2E-08 1.5E-06	3.5E-10 2.8E-08 1.4E-07		3.3E-10 1.0E-08 1.1E-06
II $h = 1/32$	1.031 1.50 2.0			Max error 1.8E-07	6.3E-09 2.7E-07 2.5E-08

Note. It is known that scientists have fully investigated the numerical solutions of ordinary differential equations because they wished to solve integral and integro-differential equations by applying the methods of differential equations. For the sake of demonstration, suppose that the kernel is degenerate and has the following form:

$$K(x,s,y) = a(x)b(s,y)\,. \tag{44}$$

In this case, one can rewrite the problem (1) as follows:

$$y' = f(x,y) + a(x)v(x), \quad y(x_0) = y_0, \tag{45}$$

$$v'(x) = b(x,y), v(x_0) = 0. \tag{46}$$

Thus, one can replace solving problem (1) with solving problems (45) and (46), which are initial-value problems that can be solved by the methods of ordinary differential equations.

It is known that the problem encapsulated by (45) and (46) consists of two ODEs of the first order. Unfortunately, this simplification is not always correct. Indeed, from the derivative of (44) we obtain

$$y'' = \frac{df(x,y(x))}{dx} + a'(x)v(x) + a(x)b(x,y).$$

The equation obtained above is an integro-differential equation. In the considered examples, the kernels have the form (44).

4. Conclusion

In this paper, some information on the solving of integro-differential equations is given. We have investigated solving an initial-value problem in the class of Volterra integro-differential equations using hybrid methods, starting with the work of Makroglou (see [16]). Note that the constructed hybrid methods are symmetric. However, asymmetric hybrid methods are usually more accurate than symmetric ones. We have constructed an asymmetric stable hybrid method with degree $p = 9$ for the case $k = 2$. However, the application of this method to practical problems is more difficult than using symmetric methods. It is impossible to investigate all aspects of this problem in a single paper. We believe that the proposed method will have many applications in the future. Note that to use (26)-(28) or (40)-(43) methods to solve certain problems, one can use block methods or predictor-corrector methods. By single comparison we show that fir selected values of variable α in formula (40), one can construct methods with the different characteristic. We also show that system (33) of nonlinear algebraic equations can have a solution in the case when the quantity of equalities is greater than the quantity of the unknowns (see e.g. variant II). It should be noted that some results obtained in this paper are a modification of corresponding results obtained in [30]. In conclusion, it should be noted that if the known methods have the order of accuracy p, the methods proposed here have the order of accuracy of more that $3p$ or $5p$. Therefore, we hope that these methods will be useful for the wide classes of specialists. Some of these methods have been applied to solve the Volterra integral equations with the symmetric boundaries, in which the advantages of these methods have been shown (see [31]).

Acknowledgment

The authors wish express their gratitude to Academicians Telman Aliev and Ali Abbasov for suggesting an investigation of the computational aspects. The research is supported by the Science Development Foundation of Azerbaijan (Grant No EIF-KETPL-2-20151(25)-56/07/1).

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