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Controllability and observability of second order linear time invariant systems

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ABSTRACT

In the present paper, controllability and observability of second order linear time invariant systems in matrix form are considered. Without reducing to first order systems, we show how the classical conditions for first order linear systems can be generalized to this case. In term of Kalman criteria, these concepts are investigated for second order discrete and continuous time linear systems. By repeated differentiation of state and output vector-functions, we derive two different systems of linear algebraic equations. Then the initial values x_0, x_1 and input functions can be determined uniquely from these systems if and only if the observability and controllability matrices have full rank, respectively. The transfer function of the second order continuous-time linear state-space system is also constructed. Numerical examples are presented to illustrate the theoretical results.

This paper is dedicated to the memory of Professor Lotfi A. Zadeh (1921-2017),
Founder of Fuzzy logic and Fuzzy Mathematics

1. Introduction

This paper focuses on second order time-invariant finite-dimensional systems, covering both continuous and discrete-time topics. Second order linear systems theory has played an important role in many technology advancements, for instance, in aerospace, communications, automotive, and computer engineering, forced oscillations and resonance phenomenon, earthquake – induced vibrations of multi-story buildings, etc. The significant advantage of modern linear control theory [1, 2] over the classical theory is its applicability to control problems involving time invariant and time-varying systems.

Since the behavior of many systems of importance in engineering practice is governed by second order controllable systems, our objective in the present paper is to contribute to the analysis of the controllability and observability of linear discrete/continuous time of second order systems. We recall that controllability is an important property of a control system, and this property plays a crucial role in many control problems, such as stabilization of unstable systems by feedback, or optimal control. Controllability and observability are dual aspects of the same problem and represent two major concepts of modern control system theory. These concepts were introduced by Kalman [1]. They can be roughly defined as follows. In order to be able to do whatever we want with a given dynamic system under control input, the system must be controllable. In order to see what is going on inside the system under observation, the system must be observable. In this paper, we show that the concepts of controllability and observability are related to linear systems of algebraic equations.

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It is well known that a solvable system of linear algebraic equations has a solution if and only if the rank of the system matrix is full. Observability and controllability tests will be connected to the rank tests of certain matrices: the controllability and observability matrices.

The works [3-6, 7-19, 20] focus on different first order time-invariant and time-varying finite-dimensional systems, covering both continuous and discrete-time topics. The note [5] studies the controllability and observability for piecewise time-varying impulsive systems. Necessary and sufficient criteria for complete controllability and observability are established. The paper [8] introduces the concept of 'hybrid-dual systems' and presents some new results on the controllability and observability of these systems. It is proved that for hybrid-dual systems, which include dual systems as a special case, controllability of one implies observability of the other. The purpose of the paper [7] is to rigorously establish a duality between reachability and observability for linear impulsive systems both in direct system-theoretic terms as well as in geometric terms. These connections are established using an appropriately defined adjoint system associated with the original impulsive system. In the paper [18], several different concepts of controllability are investigated for a class of linear singular systems which system parameters are piecewise constant. Necessary and sufficient geometric criteria for special controllability of such systems are established, respectively. These conditions can be easily transformed into algebraic form. In the paper [3] continuous-time LTI systems with either unknown-input or with lack of information about the input and output derivatives are considered. The unknown-input observability subspace and the observability subspace with unknown derivatives of input and output are computed.

The remainder of this paper is organized as follows. In Section 2, we consider a linear, time invariant, second order discrete-time system in the state space form with output measurements. The higher order difference operator method applied in the present analysis is not standard, and its exposition in the framework of investigated problem is new. It is shown that using even and odd-orders finite differences and observability matrix the initial values x_0, x_1 can be determined. In Section 3 for the purpose of studying observability of second order continuous system, we consider an input-free system with the corresponding measurements and show how the classical observability conditions for first order systems can be generalized to this case. Repeatedly differentiating of output vector-function we derive two different system of linear algebraic equations. Then the initial values x_0, x_1 can be determined uniquely from these systems if and only if the observability matrix has full rank. Section 4 deals with the controllability of second order time-invariant linear systems; it is possible to find a control sequence such that the given point can be reached from any initial state in a finite time. The system (15) is controllable if and only if rank $\mathcal{C}(A, B) = n$, where n is an order of a square matrix A. Section 5 devoted to study the evolution of the concept of controllability of second order linear continuous systems. We show how to find a control input that will transfer our system from any initial state to any final state. For a vector input system $x'' = Ax + Bu$ the above discussion produces the same relation with the controllability matrix and with the input vector $u(t)$. We remarked that the observability results of Section 3 can be obtained from duality relation between observability and controllability. Numerical examples are presented to illustrate the theoretical results; we present some numerical results corresponding to different cases of checking the controllability and observability conditions. In particular, in the one example involving single-input single-output system, by zero-pole cancellations principle of the transfer function $H(s)$ [4] we show that system is observable and controllable.

2. Observability of second order discrete systems

This section is devoted to observability of controllable second order discrete systems. Criteria for determining observability for these systems are developed. Let us consider a linear, time invariant, second order discrete-time system in the state space form

$$x_{t+2} - 2x_{t+1} + x_t = Ax_t + Bu_t, x_0 = a_o, x_1 = a_1, \tag{1}$$

with output measurements

$$y_t = Cx_t, t = 0,1,2, \dots \tag{2}$$

where $x_t = (x_t^1, \dots, x_t^n) \in \mathbb{R}^n, y_t = (y_t^1, \dots, y_t^p) \in \mathbb{R}^p$, a_0, a_1 unknown vectors, A, B, C are constant matrices of appropriate dimensions; $A \in \mathbb{R}^{n,n}, B \in \mathbb{R}^{n,r}, C \in \mathbb{R}^{p,n}$. Here $\mathbb{R}^{n,l}$ denotes the vector space of $n \times l$ real matrices, x_t is the state, u_t is the unknown input vector, and y_t is the output of the system. Now using only information from the output measurements (2) we will solve the observability problem for the state space variables defined in (1).

We say that the linear discrete-time system modelled by (1) and (2) *is observable* at time $t = 0$ if there exits some $t_1 > 0$ such that the state $(x_0, x_1) = (a_0, a_1)$ at time $t = 0$ can be uniquely determined from the knowledge of $u_t, y_t, t = 0,1, \dots, t_1$.

For a known a_0, a_1, of course, the recursion (1) gives us complete knowledge about the state variables at any discrete-time instant. Thus, our main problem is to determine from the state measurements the initial state vector $x_0 = a_0, x_1 = a_1$. The developments applied in this section are not standard, and their exposition in the framework of problem (1) is new.

Obviously, the n-dimensional vector has unknown components, and we try to determine an initial values a_0 and a_1 using output measurements y_t. Take $t = 0,1,2, \dots, n-1$ in (1) and (2), and generate the following sequence

$$y_0 = Cx_0, \Delta y_0 = y_1 - y_0 = C(x_1 - x_0) = C\Delta x_0, \Delta x_0 = x_1 - x_0,$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0 = C(x_2 - 2x_1 + x_0) = C\Delta^2 x_0 = CAx_0 + CBu_0,$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0 = C(x_3 - 3x_2 + 3x_1 - x_0) = CAx_1 + CB\Delta u_0,$$
$$\Delta^4 y_0 = CA^2 x_0 + CABu_0 + CB\Delta u_0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\Delta^{2n-2} y_0 = CA^{n-1} x_0 + CA^{n-2} Bu_0 + CA^{n-3} B\Delta^2 u_0 + \dots + CB\Delta^{2n-4} u_0,$$
$$\Delta^{2n-1} y_0 = CA^{n-1} x_1 + CA^{n-2} Bu_0 + CA^{n-3} B\Delta^3 u_0 + \dots + CB\Delta^{2n-3} u_0, \tag{3}$$

where s th-order difference operator is defined as follows

$$\Delta^s x_t = \sum_{k=0}^{s} (-1)^k C_s^k x_{t+s-k}, \; C_s^k = \frac{s!}{k!\,(s-k)!}, t = 0,1,2, \dots$$

Now let us separate equations (3) involving even and odd-orders finite differences in the following relevant matrix form

$$\begin{bmatrix} y_0 \\ \Delta^2 y_0 - CBu_0 \\ \Delta^4 y_0 - CABu_0 - CB\Delta u_0 \\ \vdots \\ \Delta^{2(n-1)} y_0 - CA^{n-2} Bu_0 - CA^{n-3} B\Delta^2 u_0 - \dots - CB\Delta^{2n-4} u_0 \end{bmatrix}_{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(np)\times n} \cdot x_0;$$

$$\begin{bmatrix} \Delta y_0 \\ \Delta^3 y_0 - CB\Delta u_0 \\ \Delta^5 y_0 - CAB\Delta u_0 - CB\Delta^3 u_0 \\ \vdots \\ \Delta^{2n-1} y_0 - CA^{n-2} B\Delta u_0 - CA^{n-3} B\Delta^3 u_0 - \dots - CB\Delta^{2n-3} u_0 \end{bmatrix}_{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(np)\times n} \cdot x_1. \tag{4}$$

We know from linear algebra that each system of linear algebraic equations with n unknowns of (4), has a unique solution x_0 and x_1 if and only if the system matrix has rank n.

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix} = n. \tag{5}$$

The initial values x_0, x_1 are determined if the so-called observability matrix $\mathcal{O}(A,C)$

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}_{(np)\times n} \tag{6}$$

has rank$n$, that is

$$\text{rank}\mathcal{O}(A,C) = n. \tag{7}$$

Theorem 2.1 The second order linear discrete-time system (1) with measurements (2) is observable if and only if the observability matrix (6) has rank$n$.

Example 2.1 Consider the following second order linear discrete system with measurements

$$\begin{bmatrix} x_{t+2}^1 - 2x_{t+1}^1 + x_t^1 \\ x_{t+2}^2 - 2x_{t+1}^2 + x_t^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix},$$

$$y_t = [1 \quad 3]\begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix}; A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, C = [1 \quad 3].$$

In this case, the equations (4) are simplified as follows

$$\begin{bmatrix} y_0 \\ \Delta^2 y_0 \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x_0; \begin{bmatrix} \Delta y_0 \\ \Delta^3 y_0 \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x_1$$

and the observability matrix for this second-order system is

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 11 & 13 \end{bmatrix}.$$

The determinant of this square matrix is nonzero (det $\mathcal{O}(A,C) = -20 \neq 0$) and the rows of the matrix are linearly independent. Thus, rank $\mathcal{O}(A,C) = n = 2$ i.e. the system under consideration is observable.

Example 2.2 Suppose now the corresponding system and measurement matrices of our problem are given as follows

$$A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}, C = [1 \quad 3].$$

Obviously, the observability matrix is

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}.$$

It is not hard to see that rank $\mathcal{O}(A,C) = 1$. It means that rank $\mathcal{O}(A,C) = 1 < n$, where $n = 2$ and the system is unobservable.

3. Observability of second order continuous systems

In this section, we study an observability problem for a second order continuous-time linear state-space system

$$x''(t) = Ax(t) + Bu(t), x(0) = x_0, x'(0) = x_1 \tag{8}$$

with the corresponding measurements

$$y(t) = Cx(t) \tag{9}$$

of dimensions $x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p$ and matrices $A \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{p,n}$, x_0, x_1 unknown vectors. Here $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^r$ and $y(t) \in \mathbb{R}^p$ are respectively the state vector, the unknown input vector and the output vector. We will conclude that the knowledge of x_0, x_1 is sufficient to determine at any time instant. This difficulty is also related to the one we encountered in the previous section when dealing with the difference operator. The problem is to find x_0, x_1 from the available measurements (9). We have solved this problem for the second order discrete-time systems by defining the sequence of measurements at discrete-time instants $t = 0,1,2, \ldots, n-1$. By analogy way in the second order continuous-time system (8) by computing higher order derivatives of the continuous-time measurements (9) at a point $t = 0$ we have

$$\begin{gathered} y(0) = Cx(0) = Cx_0, y'(0) = Cx'(0) = Cx_1, \\ y''(0) = CAx_0 + CBu(0), y'''(0) = CAx_1 + CBu'(0), \\ y^{(IV)}(0) = CA^2x_0 + CABu(0) + CBu''(0), \\ y^{(V)}(0) = CA^2x_1 + CABu'(0) + CBu'''(0), \\ \ldots\ldots\ldots \\ y^{(2n-2)}(0) = CA^{n-1}x_0 + CA^{n-2}Bu(0) + CA^{n-3}Bu''(0) + \cdots + CBu^{(2n-4)}(0), \\ y^{(2n-1)}(0) = CA^{n-1}x_1 + CA^{n-2}Bu'(0) + CA^{n-3}Bu'''(0) + \cdots + CBu^{(2n-3)}(0), \end{gathered} \tag{10}$$

Equations (10) according to even and odd-orders derivatives of input and output functions comprise the following two system of np linear algebraic equations:

$$\begin{array}{cc} y(0) = Cx(0) = Cx_0, & y'(0) = Cx'(0) = Cx_1, \\ y''(0) = CAx_0 + CBu(0), & y'''(0) = CAx_1 + CBu'(0), \\ y^{(IV)}(0) = CA^2x_0 + CABu(0) + CBu''(0), & y^{(V)}(0) = CA^2x_1 + CABu'(0) + CBu'''(0), \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ y^{(2n-2)}(0) = CA^{n-1}x_0 + CA^{n-2}Bu(0) & y^{(2n-1)}(0) = CA^{n-1}x_1 + CA^{n-2}Bu'(0) \\ +CA^{n-3}Bu''(0) + \cdots + CBu^{(2n-4)}(0), & +CA^{n-3}Bu'''(0) + \cdots + CBu^{(2n-3)}(0), \end{array} \tag{11}$$

By analogy with the second order discrete-time system (11) can be rewritten in the following matrix forms:

$$\begin{bmatrix} y(0) \\ y''(0) - CBu(0) \\ y^{(IV)}(0) - CABu(0) - CBu''(0) \\ \vdots \\ y^{2n-2}(0) - CA^{n-2}Bu(0) - CA^{n-3}Bu''(0) - \cdots - CBu^{(2n-4)} \end{bmatrix}_{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(np)\times n} x_0 = \mathcal{O}x_0 \tag{12}$$

$$\begin{bmatrix} y'(0) \\ y'''(0) - CBu'(0) \\ y^{(V)}(0) - CABu'(0) - CBu'''(0) \\ \vdots \\ y^{2n-1}(0) - CA^{n-1}Bu'(0) - CA^{n-3}Bu'''(0) - \cdots - CBu^{(2n-3)} \end{bmatrix}_{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(np)\times n} x_1 = \mathcal{O}x_1, \tag{13}$$

where $\mathcal{O} = \mathcal{O}(A, C)$ is the observability matrix already defined in (6). Thus, for determining of the initial values x_0, x_1 uniquely from (12), (13) it is necessary and sufficient that the observability matrix has full rank, i.e. rank $\mathcal{O} = n$. It would be noted that taking more higher-order derivatives in (12) and (13) it is impossible to increase the rank of the observability matrix $\mathcal{O}(A, C)$, because by the Cayley-Hamilton theorem for $m \geq n$ we have $A^m = \sum_{j=0}^{n-1} \beta_j A^j$ and so the additional equations would be linearly dependent on the previously defined n equations (12).

Thus, we have proved the following theorem for the validity of Kalman criterion.

Theorem 3.1 For observability of the linear second order continuous-time system (8) with measurements (9) it is necessary and sufficient that the observability matrix has full rank.

Remark 3.1 It can be pointed out that to prove Theorem 3.1, we can use the idea of construction the discrete-approximation problem for second order continuous system (8); at first we introduce the first and second order difference operators

$$\Delta x(t) := \frac{1}{h}[x(t+h) - x(t)], \Delta^2 x(t) := \frac{1}{h}[\Delta x(t+h) - \Delta x(t)], t = 0, h, 2h, \ldots,$$

where h is a step on the t-axis and $x(t) \equiv x_h(t)$ is a grid functions on a uniform grid. We define the following discrete-approximation problem associated with the problem (18)

$$\Delta^2 x(t) = Ax(t) + Bu(t), t = 0, h, 2h, \ldots, x(0) = x_0, \Delta x(0) = \frac{x(0+h) - x(0)}{h} = x_1. \quad (14)$$

Note that by using the definition of s th-order difference operator $\Delta^s x(t)$ we have associated with the linear state-space system (8) the following discrete-approximate equation

$$\Delta^s x(t) = A\Delta^{s-2} x(t) + B\Delta^{s-2} u(t), s = 2,3, \ldots, 2n+1.$$

We proceed by analogy with the preceding derivation of Section 2; only in this case in the equations (4) for second order discrete-approximation-time system (14) instead of $\Delta^s y_0, \Delta^s u_0, x_0, x_1, y_0$ we take $\Delta^s y(0), \Delta^s u(0), x(0), \Delta x(0), y(0)$, respectively:

$$\begin{bmatrix} y(0) \\ \Delta^2 y(0) - CBu(0) \\ \Delta^4 y(0) - CABu(0) - CB\Delta^2 u(0) \\ \vdots \\ \Delta^{2(n-1)} y(0) - CA^{n-2}Bu(0) - \cdots - CB\Delta^{2n-4} \end{bmatrix}_{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(np)\times n} x(0);$$

$$\begin{bmatrix} y(0) \\ \Delta^3 y(0) - CB\Delta u(0) \\ \Delta^5 y(0) - CABu(0) - CB\Delta^3 u(0) \\ \vdots \\ \Delta^{2n-1} y(0) - CA^{n-2}B\Delta u(0) - CA^{n-3}B\Delta^3 u(0) - \cdots - CB\Delta^{2n-3} \end{bmatrix}_{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(np)\times n} \Delta x(0);$$

Then derivation of conditions (12) is implemented by passing to the limit here (at least formally), whenever the discrete steps tend to zero.

4. Controllability of second order discrete systems

We have seen earlier that controllability is an important property of a control system, and besides controllability property plays a crucial role in many control problems, such as stabilization of unstable systems by feedback, or optimal control. Now, we consider again a linear discrete-time invariant control system defined in Section 2:

$$x_{t+2} - 2x_{t+1} + x_t = Ax_t + Bu_t, x_0 = a_0, x_1 = a_1 \quad (15)$$

Thus, the linear second order discrete-time system modelled by (15) *is controllable* if it is possible to find a control sequence such that the given point can be reached from any initial state in a finite time. In other words, we want to transfer our system from any initial state $x_0 = a_0, x_1 = a_1$ to any desired final state $x_{t_1} = x_{0f}, \Delta x_{t_1} = x_{1f} - x_{0f}$ in a finite time $t_1 < \infty$.

By using the definition of s th-order difference operator $\Delta^s x_t$, we have

$$\Delta^s x_t = A\Delta^{s-2} x_t + B\Delta^{s-2} u_t, s = 2,3, \ldots, 2n+1.$$

Then we continue our techniques used in the study of observability problems of the preceding sections; by repeatedly substitutions here we derive the following set of equations

$$\begin{gathered} \Delta^2 x_0 = Ax_0 + Bu_0, \ \ \Delta^3 x_0 = A\Delta x_0 + B\Delta u_0, \\ \Delta^4 x_0 = A\Delta^2 x_0 + B\Delta^2 u_0 = A^2 x_0 + ABu_0 + B\Delta^2 u_0, \\ \Delta^5 x_0 = A\Delta^3 x_0 + B\Delta^3 u_0 = A^2 \Delta x_0 + AB\Delta u_0 + B\Delta^3 u_0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta^{2n} x_0 = A^n x_0 + A^{n-1}Bu_0 + A^{n-2}\Delta^2 u_0 + \cdots + B\Delta^{2n-2} u_0, \\ \Delta^{2n+1} x_0 = A^n \Delta x_0 + A^{n-1}B\Delta u_0 + A^{n-2}\Delta^3 u_0 + \cdots + B\Delta^{2n-1} u_0. \end{gathered} \quad (16)$$

In a general case, as earlier, according to even and odd-orders finite derivatives when the input

u_t is a vector of dimension r, splitting the equations (16) on two group, the repetition of the same procedure give us

$$\Delta^{2n}x_0 - A^n x_0 = [B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-2}B \vdots A^{n-1}B]\begin{bmatrix}\Delta^{2n-2}u_0\\ \vdots \\ \Delta^2 u_0 \\ u_0\end{bmatrix} \tag{17}$$

$$\Delta^{2n+1}x_0 - A^n x_1 = [B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-2}B \vdots A^{n-1}B]\begin{bmatrix}\Delta^{2n-1}u_0\\ \vdots \\ \Delta^3 u_0 \\ \Delta u_0\end{bmatrix} \tag{18}$$

Where $[B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ is a matrix of dimension $n \times (r \cdot n)$. In what follows we denote it by $\mathcal{C}(A,B)$ define it as the controllability matrix:

$$\mathcal{C}(A,B) = [B \vdots AB \vdots \cdots \vdots A^{n-1}B]. \tag{19}$$

The system of n linear algebraic equations in $r \cdot n$ unknowns for n number of r-dimensional vector components of $u_0,\ \Delta^2 u_0, \dots, \Delta^{2n-2}u_0$ and $\Delta u_0,\ \Delta^3 u_0, \dots, \Delta^{2n-1}u_0$, are the following equations

$$\mathcal{C}^{n\times(n\cdot r)}(A,B)\begin{bmatrix}\Delta^{2n-2}u_0\\ \vdots \\ \Delta^2 u_0 \\ u_0\end{bmatrix}_{(nr)\times 1} = x_{0f} - A^n x_0;\ \ \mathcal{C}^{n\times(n\cdot r)}(A,B)\begin{bmatrix}\Delta^{2n-1}u_0\\ \vdots \\ \Delta^3 u_0 \\ \Delta u_0\end{bmatrix}_{(nr)\times 1} = x_{1f} - A^n x_1.$$

Then these equations will have a solutions for any given pair of vectors x_{0f}, x_{1f} if and only if the matrix has full rank, i.e. rank $\mathcal{C}(A,B) = n$. These results are summarized in the following theorem.

Theorem 4.1 The linear discrete-time system (15) is controllable if and only if rank $\mathcal{C}(A,B) = n$, where the controllability matrix is defined by (19).

5. Controllability of second order continuous systems

As is known, the basic concepts of controllability with continuous systems play an essential, fundamental role in the solutions of many important different optimal control problems. Roughly speaking, as in the discrete systems we remind that controllability of continuous systems generally means that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

Suppose we have a simplified problem, where the input is a scalar and input matrix B is a vector, usually denoted by b:

$$x'' = Ax + bu, x(0) = x_0, x'(0) = x_1. \tag{20}$$

A system is said to be (state) controllable at time $t = 0$, if there exists a finite $t_1 > 0$ such that for any x_0 and any x_1, there exist an input $u(t), t \in [0,1]$ that will transfer the state $(x(0), x'(0))$ to the state $(x(t_1), x'(t_1))$ at time t_1, otherwise the system is said to be uncontrollable at time $t = 0$.

By sequential differentiation of equation (20), we derive the following set of equations

$$x'' = \frac{d^2x}{dt^2} = Ax + bu,\quad x''' = \frac{d^3x}{dt^3} = Ax' + bu',$$
$$x^{(IV)} = \frac{d^4x}{dt^4} = Ax'' + bu'' = A^2x + Abu + bu'',$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..$$
$$x^{(2n)} = \frac{d^{2n}x}{dt^{2n}} = A^n x + A^{n-1}bu + A^{n-2}bu'' + \cdots + bu^{(2n-2)},$$
$$x^{(2n+1)} = \frac{d^{2n+1}x}{dt^{2n+1}} = A^n x' + A^{n-1}bu' + A^{n-2}bu''' + \cdots + bu^{(2n-1)}. \tag{21}$$

Notice that in this case the controllability matrix

$$\mathcal{C}(A,b) = [b \vdots Ab \vdots A^2b \vdots \cdots \vdots A^{n-2}bA^{n-1}b]$$

is a square matrix of size $n \times n$. If the controllability matrix is non-singular, i.e. its determinant is nonzero, then equation (21) has the unique solution for the input sequence given by

$$\mathcal{C}^{n\times n}(A,b)\begin{bmatrix} u^{(2n-2)}(t) \\ u^{(2n-4)}(t) \\ \vdots \\ u''(t) \\ u(t) \end{bmatrix} = x^{(2n)}(t) - A^n x(t);\ \mathcal{C}^{n\times n}(A,b)\begin{bmatrix} u^{(2n-1)}(t) \\ u^{(2n-3)}(t) \\ \vdots \\ u'''(t) \\ u'(t) \end{bmatrix} = x^{(2n+1)}(t) - A^n x'(t). \qquad (22)$$

Note that (22) is valid for any $t \in [0, t_f]$ with an arbitrary finite t_f. Thus, the nonsingularity of the controllability matrix implies the existence of the scalar input function $u(t)$ and its $2n-1$ derivatives, for any $t < t_f < \infty$.

For a vector input system $x'' = Ax + Bu$ where B is $n \times r$ matrix, the above discussion produces the same relation as (22) with the controllability matrix given by (19) and with the input vector $u(t) \in \mathbb{R}^n$. It is well known from linear algebra that a solution of (22) exists for any $x^{(2n)}(t) - A^n x(t)$ and $x^{(2n+1)}(t) - A^n x'(t)$ and any desired state at t if and only if rank $\mathcal{C}(A,b) = n$.
We have obtained the following theorem.

Theorem 5.1 The linear continuous-time system is controllable if and only if the controllability matrix has full rank, i.e. rank $\mathcal{C}(A,b) = n$.

Remark 5.1 We recall that these observability results can be obtained from duality relation between observability and controllability (see [7] and references therein).

Indeed, by this relation $x'' = Ax + Bu, y = Cx$ is observable if and only if the dual system $x^{*\prime\prime} = -A^T x^* + C^T u$ with output vector $z = B^T x^*$ is controllable, where a prime denotes the adjoint transformation. But according to the Theorem 4.1 for controllability of the last dual system (with respect to the state x^*) the controllability $n \times (np)$ matrix $[C^T \vdots A^T C^T \vdots (A^2)^T C^T \vdots \cdots \vdots (A^{n-1})^T C^T]$ should be have full rank.

Example 5.1. Suppose we have the linear continuous-time system

$$x'' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 4 & -5 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} u.$$

Here the matrix B is vector-column b of dimension three and the controllability matrix for this third-order system is given by

$$\mathcal{C}(A,b) = [b \vdots Ab \vdots A^2 b] = \begin{bmatrix} 1 & -2 & -12 \\ 0 & 5 & 21 \\ 3 & 10 & -13 \end{bmatrix}.$$

Since det $\mathcal{C}(A,b) = -211 \neq 0$ we can conclude that rank $\mathcal{C}(A,b) = 3$. Thus, rank $\mathcal{C}(A,b) = 3 = n$ implies that the system under consideration is controllable.

The following result is the bridge between the transfer functions of first and second order linear systems with continuous-time linear state-space system and with the corresponding measurements.

Lemma 5.1 The transfer function of the second order continuous-time linear state-space system $x'' = A_1 x + Bu$ with the corresponding measurements $y = Cx$ has the form $H(s) = C[s^2 E - A_1]^{-1} B$.

Proof. Introducing a new variable $v = \begin{bmatrix} x \\ x' \end{bmatrix}$ it is easy to see that our system can be converted to the first order system $v' = \tilde{A} v + \tilde{B} u$, where $\tilde{A} = \begin{bmatrix} 0_{n\times n} & E_n \\ A_n & 0_{n\times n} \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 0_{n\times r} \\ B \end{bmatrix}$ ($0_{n\times r}$ is the $n \times r$ zero matrix). By analogy $y = [C \quad 0_{p\times n}] v$. Then for a new problem the transfer function [21] is $H(s) = [C \quad 0_{p\times n}][sE_{2n} - \tilde{A}]^{-1} \tilde{B}$ (E_k is $k \times k$ identity matrix). Now it is not hard to compute the inverse matrix of the invertible matrix $[sE_{2n} - \tilde{A}]$ partitioned into a block form:

$$\left[sE_{2n}-\tilde{A}\right]^{-1}=\begin{bmatrix} sE_n & -E_n \\ -A_1 & sE_n \end{bmatrix}^{-1}=\begin{bmatrix} \left[sE_n-\frac{1}{s}A_1\right]^{-1} & [s^2E_n-A_1]^{-1} \\ [s^2E_n-A_1]^{-1} & \left[sE_n-\frac{1}{s}A_1\right]^{-1} \end{bmatrix}. \quad (23)$$

Then substitution (23) into the transfer function we have immediately

$$H(s)=C[s^2E-A_1]^{-1}B, E=E_n.$$

Example 5.2 Consider the following system with scalar input and measurements

$$x''=\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}x+\begin{bmatrix} 1 \\ 2 \end{bmatrix}u; y=[1 \quad 3]x; A=\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}; b=\begin{bmatrix} 1 \\ 2 \end{bmatrix}; C=[1 \quad 3].$$

Let $H(s)=C[s^2E-A]^{-1}b$ be the transfer function [21] of this single-input single-output system, where E is 2×2 identity matrix. Then

$$[s^2E-A]^{-1}=\begin{bmatrix} s^2-2 & -1 \\ -3 & s^2-4 \end{bmatrix}^{-1}=\begin{bmatrix} \frac{s^2-4}{(s^2-2)(s^2-4)-3} & \frac{1}{(s^2-2)(s^2-4)-3} \\ \frac{3}{(s^2-2)(s^2-4)-3} & \frac{s^2-2}{(s^2-2)(s^2-4)-3} \end{bmatrix}$$

and

$$H(s)=C[s^2E-A]^{-1}b=[1 \quad 3]\begin{bmatrix} \frac{s^2-4}{(s^2-2)(s^2-4)-3} & \frac{1}{(s^2-2)(s^2-4)-3} \\ \frac{3}{(s^2-2)(s^2-4)-3} & \frac{s^2-2}{(s^2-2)(s^2-4)-3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Consequently, $H(s)=\frac{7s^2-5}{s^4-6s^2+5}$ and there is no zero-pole cancellations in the transfer function of our a single-input single-output system, that is the system is both controllable and observable (i.e., the poles are $\pm\sqrt{5}$; ± 1 and zero is $\pm\sqrt{5/7}$). Notice that systems that satisfy this relationship are called proper. To get the complete answer we have to go to a state space form and examine the controllability and observability matrices; in Example 2.1 it was already shown that rank$\mathcal{O}(A,C)=n=2$ and the system is observable (we recall that Examples 2.1 and 5.2 have the same matrices A,C). On the other hand, $\mathcal{C}(A,b)=[b \vdots Ab]=\begin{bmatrix} 1 & 4 \\ 2 & 11 \end{bmatrix}$, i.e., det $\mathcal{C}(A,b)=3\neq 0$ and our system is controllable.

6. Conclusion

In this paper, the issue of the observability and controllability criteria for a class of second order linear time invariant systems is addressed. Several sufficient and necessary conditions for observability and controllability of such systems are established in the form of Kalman's type conditions. By sequential differentiation of state and output vector-functions with scalar and vector inputs deriving different systems of linear algebraic equations are also discussed, transfer function is constructed. It is worth noticing that this note considers the controllability and observability for a class of linear time invariant systems. Many issues are still untouched on more general linear systems, for instance, higher order linear systems with variable coefficients. Numerical examples are presented to illustrate the theoretical results and to show the effectiveness of the proposed results.

References

[1] R.E. Kalman, On the General Theory of Control Systems, IFAC Proceed. 1 (1960) 491–502. https://doi.org/10.1016/S1474-6670(17)70094-8

[2] L.A. Zadeh, C.A.Desoer, Linear System Theory: The State Space Approach, McGraw-Hill Series in System Science, New York, NY: McGraw-Hill, 1963.

[3] N. Eskandari, Meeko M. K. Oishi, Z. Jane Wang, Observability Analysis of Continuous-Time LTI Systems with Limited Derivative Data, Asian J. Contr. 16 (2013) 623–627. https://doi.org/10.1002/asjc.749

[4] H.F. Chen, On stochastic observability and controllability, Automatica 16 (1980) 179–190. https://doi.org/10.1016/0005-1098(80)90053-9

[5] J. Han, Y. Liu, S. Zhao, R. Yang, A Note on the Controllability and Observability for Piecewise Linear Time-varying Impulsive Systems, Asian J. Contr. 15 (2012) 1563–1876. https://doi.org/10.1002/asjc.642

[6] X. Jiang, X. Tian, W. Zhang, Exact observability canonical forms of stochastic time-invariant systems based on $\mathscr{H}$-representation technique, Proceeding of the 2015 IEEE International Conference on Information and Automation Lijiang, China, August (2015) 1009–1013. 10.1109/ICInfA.2015.7279434

[7] D.A. Lawrence, Duality Properties of Linear Impulsive Systems, Asian J. Contr. 14 (2012) 901–910. https://doi.org/10.1002/asjc.513

[8] P.M. Lin, Y.L. Kuo, Some new results on the controllability and observability of linear systems and networks, Inter. J. Circuit Theory Appl. 7 (1979) 413–422. https://doi.org/10.1002/cta.4490070406

[9] L. Zhang, D. Cheng, J.B. Liu, Stabilization of Switched Linear systems, Asian J. Contr. 5 (2008) 476–483. https://doi.org/10.1111/j.1934-6093.2003.tb00165.x

[10] R. Ortega, Z. Liu, H.Su, Control via interconnection and damping assignment of linear time-invariant systems: a tutorial, Intern. J. Contr. 85 (2012) 603–611. https://doi.org/10.1080/00207179.2012.660734

[11] E.N. Mahmudov, Approximation and Optimization of Discrete and Differential Inclusions, Boston: Elsevier, 2011.

[12] E.N. Mahmudov, Transversality condition and optimization of higher order ordinary differential inclusions, Optimization 64 (2015) 2131–2144. https://doi.org/10.1080/02331934.2014.929681

[13] E.N. Mahmudov, Optimization of Second-Order Discrete Approximation Inclusions, Numeric. Funct. Anal. Optim. 36 (2015) 624–643. https://doi.org/10.1080/01630563.2015.1014048

[14] E.N. Mahmudov, Optimization of Fourth Order Sturm-Liouville Type Differential Inclusions with Initial Point Constraints, J. Indust. Manag. Optim. (JIMO) (2018) 13–35. https://doi.org/10.3934/jimo.2018145

[15] E.N. Mahmudov, Optimal Control of Second Order Delay-Discrete and Delay-Differential Inclusions with State Constraints, Evolut. Equat. Contr. Theory (EECT) 7 (2018) 501–529. https://doi.org/10.3934/eect.2018024

[16] E.N. Mahmudov, Optimization of Mayer Problem with Sturm-Liouville Type Differential Inclusions, J. Optim. Theory Appl. 177 (2018) 345–375. https://doi.org/10.1007/s10957-018-1260-2

[17] M. Malisoff, L. Rifford, E. Sontag, Global Asymptotic Controllability Implies Input-to-State Stabilization, SIAM J. Control Optim. 42 (2004) 2221–2238. https://doi.org/10.1137/S0363012903422333

[18] G. Xie, L. Wang, Controllability of a Class of Singular Systems, Asian J. Contr. 8 (2008) 424–432. https://doi.org/10.1111/j.1934-6093.2006.tb00295.x

[19] D.D. Vecchio, R.M. Murray, Observability and local observer construction for unknown parameters in linearly and nonlinearly parameterized systems, Proceedings of the 2003 American Control Conference, 4-6 June, (2003) 748–4753. https://doi.org/10.1109/ACC.2003.1242473

[20] H. Zhang, Y. Tian, L. Gao, Stochastic Observability of Linear Systems under Access Constraints, Asian J. Contr. 17 (2014) 64–73. https://doi.org/10.1002/asjc.1035

[21] K.J. Astrom, R.M. Murray, Feedback Systems: An Introduction for Scientists and Engineers, Princeton University Press, Princeton and Oxford, 2008.