Informatics and Control Problems 39 Issue 2 (2019)
journal homepage: www.icp.az

On some connection between the definite and indefinite integrals

M.N. Imanova*

Science Development Foundation under the President of the Republic of Azerbaijan, Baku, Azerbaijan
Institute of Control Systems of ANAS, Baku, Azerbaijan

ARTICLE INFO

Article history:
Received 05.12.2019
Received in revised form 16.12.2019
Accepted 23.12.2019
Available online 30.12.2019

Keywords:
Numerical method
Single and double definite integrals
Indefinite integrals
Hybrid and stepwise methods
Initial-value problem
Nonlinear system of algebraic equations

ABSTRACT

It is known that in the construction of the numerical methods for solving of initial-value problem of ODE in used the methods, which have applied to calculation of the definite integrals. However, here for the calculation of definite integrals have proposed to use the methods that have used in solving of initial-value problem for the ODEs. The definite integrals are expressed by the indefinite integrals that are the solution of the above-mentioned problem. For the construction of more exact methods of calculation of the definite integrals here proposed to use forward-jumping (advanced) and hybrid methods. Have established some connection between the Gauss and hybrid methods. And have determined some necessary conditions for the satisfying of the coefficients of the proposed methods. Constructed the stable methods with the degree $p \le 8$. Have been shown that how received here results can be transferred to double definite integrals. For this aim, determines some connection between double integrals and single definite integrals. By using this relation have constructed methods that are applied to calculate the double integrals. Advantages of this method illustrated by calculation of model double integral by the constructed here methods.

1. Introduction

As you know, using certain integrals, scientists investigated many practical problems, such as calculating the area limited by certain functions or straight lines, the volume of various figures, the volumes of rotating objects, the distances between objects, and the energy of signals. earthquakes and others (see e.g. [1, pp. 169-222; 2; 3]. The study of definite integrals is closely related to the determination of the solution to the initial problem for ODEs. Therefore, many well-known scientists, such as Newton, Kottes, Gauss, Chebyshev, Simpson, Adams, etc., were involved in constructing methods for calculating certain integrals [4-7]. Scientists have built various formulas for calculating definite integrals with different accuracy.

It is known that to obtain more accurate results in the calculation of certain integrals, a decrease in the step size is used, which leads to an increase in the volume of calculation work. And it is known that the first direct method for solving the initial problem for ODEs was constructed by Euler using the calculation of definite integrals. Given that the accuracy order for the Euler method is 1 (one), therefore, to construct more accurate methods than the Euler methods, experts proposed

* *E-mail addresses:* imanovamehriban77@gmail.com (M.N.Imanova)

www.icp.az/2019/2-04.pdf

using interpolation polynomials with high accuracy in constructing methods for calculating definite integrals [6-10]. But here, to build more accurate methods, using the method of unknown coefficients is proposed. And for the study of definite integrals, the sub-interval method is used, by which the results obtained here were compared with the known.

Now let us compute definite integrals written as:

$$I = \int_{x_0}^{b} f(x)dx, \tag{1}$$

here sufficiently smooth function $f(x)$ is defined in the interval $[x_0, b]$. For the construction of the methods to calculation of definite integral (1) let us denote by f_i the values of the function of $f(x)$ at the node points $x_i = x_0 + ih (i = 0,1,..., N)$. Here $0 < h$-step size by which the interval $[x_0, b]$ to divided to N parts.

For the calculation of integral (1), let us consider the following function:

$$y(x) = \int_{x_0}^{x} f(s)ds, \quad x_0 \le x \le b. \tag{2}$$

From here, we obtain that $I = y(b)$.

It is evident that $y'(x) = f(x), y(x_0) = 0$. This is the initial-value problem for the first-order ODE. This problem in the subinterval $[x_i, x_{i+1}]$ can be written as:

$$y'(x) = f(x), \quad y(x_i) = y_i, \quad x \in [x_i, x_{i+1}]. \tag{3}$$

Let the problem to be presented as the following:

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(s)ds, (i = 0,1,..., N-1). \tag{4}$$

By using this equality, scientists constructed the methods for solving problem (3) with the different orders of accuracy. Here, we propose to use finite-difference methods applied them to solve problems (3) and (4) and also construct the methods of stepwise (advanced) and hybrid types to calculate definite integral (1).

2. Construction of finite-difference methods

As is known, one of the popular methods for solving of the problem (3) is the finite difference or multistep method, which can be presented as the following:

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} (n = 0,1,..., N-l; \quad l = \max(s,k)). \tag{5}$$

This method has been investigated by some authors [8-14].

Note that method (5) in the case $s = k$ was fundamentally investigated by G. Dahlquist [8], who showed that if the method of (5) stable in the case $s = k$ and have the degree p then $p \le 2[k/2] + 2$ and for all the values of the order k, there are stable methods with the maximum degrees. Here, we use the known definition of the conception degree and stability proposed by Dahlquist [8].

Definition 1. The method (5) is stable if the roots of the characteristic polynomials $\rho(\lambda)$ of the method (5) lie in the unit circle on the boundary of which there is no multiply root. Here $\rho(\lambda)$ can be presented as:

$$\rho(\lambda) = \alpha_s \lambda^s + \alpha_{s-1}\lambda^{s-1} + ... + \alpha_1 \lambda + \alpha_0. \tag{6}$$

Definition 2. The method has the degree of p, if the following holds:

$$\sum_{i=0}^{s}\alpha_i y(x+ih) - h\sum_{i=0}^{k}\beta_i y'(x+ih) = O(h^{p+1}), h \to 0.$$

Here p is integer size.

It follows to remark that the all known methods of type (5) subject to Dahlquist law. But in the work [13] have proved the existence of the stable methods of type (5) with the degree $p = k + m + 1$ (for the values $k \geq 3m$). The methods of the same property have constructed by other authors [14-16]. By taking into account this property of the stepwise methods here consider the application of the stepwise methods to solving of the problem (3). Therefore, suppose that $s < k$. Note that if in this case, the function of $f(x)$ depends on the function of $y(x)$, it is to say that $f(x) \equiv \varphi(x, y)$ then application of the stepwise methods accompanied by some difficulties. By taking this into account, we get that the application of the stepwise methods to calculation of the definite integrals is preferable to the known implicit or explicit methods. But for the calculation of $y(x_N)$ by stepwise methods, we need to use some values of function $f(x)$ outside of the considering segment.

By using the bounders $p \leq k + m + 1 (m \leq [k/3])$, we get that for the construction of more exact stable methods the value of k must be chosen greater. For instance, if $k \geq 9$, then we can construct the stable method of type (5) with the degree $p \leq k + 4$. As is known for increasing the values of k corresponds to become greater of the number of initial-values $y_j (j = 0,1,..., k-1)$. Therefore, scientists discussed to use the other scheme for the construction of the stable methods with the high degrees. For this aim, it is proposed to use the Gauss, Chebyshev, Labbotto, and others interpolation polynomials. As a result, new methods have appeared. These methods are left over from hybrid methods.. These methods in simple form can be written as following (see for example [17-20]:

$$\sum_{i=0}^{s}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i+\nu_i}, \tag{7}$$

here $\nu_i (i = 0,1,..., k)$ are the hybrid points, which can be defined as the solution of the some nonlinear system of algebraic equations.

Here, to construct more accurate methods, it is proposed to use the following method, which was investigated and applied to solve some similar problems [21-24]:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\gamma_i f_{n+i+\nu_i} \quad (|\nu_i| < 1, i = 0,1,..., k). \tag{8}$$

This method can be obtained by using some interpolation polynomials in the calculation of definite integrals. But here we want to build methods of type (8) using the method of unknown coefficients. For this aim let us use the equality $y'(x) = f(x)$ and consider following Taylor expansions:

$$y(x+ih)=y(x)+ihy'(x)+\frac{(ih)^2}{2!}y''(x)+...$$
$$+\frac{(ih)^p}{p!}y^{(p)}(x)+O(h^{p+1}),$$
$$y'(x+l_ih)=y'(x)+l_ihy''(x)+\frac{(l_ih)^2}{2!}y'''(x)+...$$
$$+\frac{(l_ih)^{p-1}}{(p-1)!}y^{(p)}(x)+O(h^p).$$

By using these equalities in the asymptotic equality of (4), we obtain the following conditions for the satisfaction of the coefficients $\alpha_i (i=0,1,...,k-m), \beta_i, \gamma_i, \nu_i (i=0,1,...,k)$:

$$\sum_{i=0}^{k-m}\alpha_i=0; \quad \sum_{i=0}^{k}\beta_i+\sum_{i=0}^{k}\gamma_i=\sum_{i=0}^{k-m}i\alpha_i;$$
$$\sum_{i=0}^{k}\left(\frac{i^l}{l!}\beta_i+\frac{(i+\nu_i)^l}{l!}\gamma_i\right)=\sum_{i=0}^{k-m}\frac{i^{l+1}}{(l+1)!}\alpha_i \qquad (9)$$
$$l=1,2,...,p-1.$$

By using the solution of nonlinear system (9) we can construct the stable methods of type (8). Note that in the system of (9) there are $p+1$ equations and $4k+3-m$ unknowns. If the system (9) has the solution in the case $p+1=4k+3-m$, then we get that there exist the methods of type (8) with the degree $p=4k+2-m$. Usually methods with the degree $p=4k+2-m$ are unusable for $k>2$. Here we prove that there exist stable methods with the degree $p\le 3k+2+m$. But it does not follow from here that $P_{\max}=3k+2+m(k\ge 3m)$. System (8) is nonlinear, therefore finding some conditions for the existence of the unique solution is not easy.

Therefore, the existence of stable methods of type (8) with maximum degrees is unknown to us. These properties of a nonlinear system of algebraic equations were encountered in the study of the Gauss and Chebyshev methods. Therefore, to find the Gaussian nodal points, the authors used well-known polynomials [4, p.189-199; 25, p.463-469]. But here we used the Mathcad program to find a solution to system (9) and in some cases we obtain the existence of a solution to system (9) in the case $p>3k+2+m$. Let us note that these solutions are approximate therefore asserting the existence of stable methods with the degree $p>3k+2+m$ is incorrect. But saying that there are no stable methods with the degree $p>3k+2+m$ is incorrect as well. If we put $\nu_i=0 (i=0,1,...,k)$ in the equality of (8), then we get the fundamentally investigated method which coincides with the method of (5). In this case for finding the values of the coefficients of this method, we can use the following way [26], [27]:

$$\alpha_0=-\sigma_0^{(1)}+\sigma_1^{(1)}-\sigma_2^{(1)}+...+(-1)^{k-1}\sigma_{k-2}^{(1)}+(-1)^k\sigma_{k-1}^{(1)},$$
$$\alpha_i=\sum_{j=i-1}^{k-1}(-1)^{j-i+1}(j+1)j(j-1)...(j-i+2)\sigma_j^{(1)}/i!$$
$$(i=1,2,...,k), \qquad (10)$$
$$\beta_0=\sigma_0^{(2)}-\sigma_1^{(2)}+\sigma_2^{(2)}+...+(-1)^{k-1}\sigma_{k-1}^{(2)}+(-1)^k\sigma_k^{(2)},$$
$$\beta_i=\sum_{j=i-1}^{k-1}(-1)^{j-i}j(j-1)...(j-i+1)\sigma_j^{(2)}/i!$$
$$(i=1,2,...,k),$$

For determining the values of the constant $\sigma_j^{(1)}$ and $\sigma_j^{(2)}(i=0,1,...,k)$, we can use the following system:

$$\sum_{i=0}^{j} C_i \sigma_{j-i}^{(1)} = \sigma_j^{(2)} (j=0,1,...,k; \sigma_k^{(1)}=0),$$

here

$$C_m = \sum_{\nu=1}^{m} (-1)^{\nu-1} C_{m-i} /(i+1); (C_0 = 1; m=1,2,3,...)$$

$$\sum_{i=j-k+1}^{j} C_i \sigma_{j-i}^{(1)} = 0 \qquad (j=k+1, k+2,..., p-1), \tag{11}$$

As it follows from here, determining the solution of systems (10) and (11) is much simpler than determining the solution of system (9). Because in solving (11) in the first step we find the values of constants $\sigma_0^{(1)}, \sigma_1^{(1)}, \dots, \sigma_{k-1}^{(1)}$ and after them on the second step of the values of the constant $\sigma_j^{(2)}(i=0,1,...,k)$. But in solving of the system we must find the values of $2k$-unknowns as the solution of system (9). Note that this idea is more effective when applied to construct multistep methods of the second derivative.

3. Application of the multistep second derivative methods to calculation of the definite integral

The multistep second derivative methods in more general form can be written as [16]:

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{l} \gamma_i y''_{n+i}, \tag{12}$$

here k, s, l are integer variable and independent on each other. Choosing these variables from method (12), we can obtain explicit, implicit, and stepwise methods. Therefore, method (12) is called a more general form.

Let us consider application of method (12) to calculation of integral (2). In this case, we get the following:

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{l} \gamma_i f'_{n+i}. \tag{13}$$

By taking into account that the function of $f(x)$ is known then we get that by using method (13) we can calculate the value of definite integral (1). Method (13) has been fundamentally investigated by some authors [28-31]. If method (13) is stable and has the degree of p then there exist methods with the degree $p \le 2k+2$ (in the case $s=k=l$) and there exist methods with the degree $p > 2k+2$ in other cases [16].

It follows that the concept of stability for method (13) depends on the values of the coefficients $\beta_i (i=0,1,...,k)$. If the conditions $|\beta_0|+|\beta_1|+...+|\beta_s| \ne 0$ hold, then the concept of stability for method (13) is defined as the stability of method (5). But in the case $|\beta_0|+|\beta_1|+...+|\beta_s| = 0$ method (13) is called stability if the roots of the polynomial $\rho(\lambda)$ lie in the unit circle on the boundary of which there are no multiple roots without the double root $\lambda=1$. It is evident that in this case the degree of method (13) will be same as the degree of method (5). For the simplicity let us put $s=k=l$ in equality (13). As was noted above, in this case there are stable methods with the degree $p=2k+2$ if $|\beta_0|+|\beta_1|+...+|\beta_s| \ne 0$ holds. It is easy to understand that by using method (13) we can determine the value of y_{n+k}, if the values $y_j (j = 0,1,...,k-1)$ are known. For solving this, we propose using the following method, which has been investigated in [32]:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=1}^{s} h^j \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}^{(j)}. \tag{14}$$

From this formula we can obtain the Taylor expansion for the value $k = 1$. In this case we get the one-step method which can be always applied to the calculation of definite integral (1), so that $y_0 = 0$. Thus, by using the above method, we can construct the predictor-corrector methods with the different degrees for calculation of the values $y_j (j = 0,1,...,k-1)$ and for the values $y_{n+k} (n = 0,1,..., N-k; x_N = x_0 + Nh)$.

For the construction of stable methods of type (13) (in the case $s = k = l$), let us consider for certain values of the coefficients of method (13).

As was noted above, for determining the values of the coefficients of method (13) we can use the method of unknown coefficients. And in this case, we obtain the system of algebraic equations which is similar to system (9). But here we want to present a way of finding the coefficient values for method (13), when using which the amount of computational work is less than the computational work obtained when solving a system similar to system (9).

To find the values of the coefficients $\alpha_i, \beta_i, \gamma_i (i = 0,1,...,k)$, we here propose using the following sequence of linear systems of algebraic equations and system (10):

$$\begin{aligned} &\gamma_0 = \sigma_0^{(3)} - \sigma_1^{(3)} + \sigma_2^{(3)} + ... + (-1)^{k-1}\sigma_{k-1}^{(3)} + (-1)^k \sigma_k^{(3)}; \\ &\gamma_i = \sum_{j=i-1}^{k-1} (-1)^{j-i} j(j-1)...(j-i+1)\sigma_j^{(3)} / i!; \qquad i = 1,2,...,k. \end{aligned} \tag{15}$$

But to find the values of coefficients $\sigma_m^{(l)} (l = 1,2,3; m = 0,1,...,k)$, we here propose using the following system of algebraic equations:

$$\begin{aligned} &\sum_{i=0}^{j} C_i \sigma_{j-i}^{(1)} + \sum_{i=0}^{j} \frac{(-1)^{j-i+1}}{j-i+1} \sigma_{j-i}^{(3)} = \sigma_j^{(2)}, \quad j = 0,1,...,k, \quad \sigma_k^{(1)} = 0; \\ &\sum_{i=j+1}^{j+k} C_i \sigma_{j+k-i}^{(1)} + \sum_{i=j}^{j+k} \frac{(-1)^i}{i} \sigma_{j+k-i}^{(3)} = 0, \quad j = 1,...,k; \\ &\sum_{i=j+k+1}^{j+2k} C_i \sigma_{j+2k-i}^{(1)} + \sum_{i=j+k}^{j+2k} \frac{(-1)^i}{i} \sigma_{j+2k-i}^{(3)} = 0, \quad j = 1,...,k; \\ &\sum_{i=2k+1}^{3k+1} C_i \sigma_{3k+1-i}^{(1)} + \sum_{i=2k+1}^{3k+1} \frac{(-1)^i}{i} \sigma_{3k+1-i}^{(3)} = C. \end{aligned} \tag{16}$$

Here, the constant C is the coefficient of the principal part in the asymptotic expansion of the remainder term of the constructed methods. The equivalents of system (9) and (16) can be found in [25].

Let us note that in the case $k = 2$ linear part of method (13) can be proposed as: $y_{n+2} - y_n$. In this case by replacing h with $h/2$ from the mentioned linear part, we get: $y_{n+1} - y_n$. Thus, we have constructed the one-step method, which is similar to hybrid methods. This scheme can be generalized in the following way:

$$y_{n+k} - y_n = h\sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{l} \gamma_i f'_{n+i}. \tag{17}$$

To obtain a one-step method from method (17), it is enough to choose the step size h in the form h/k. It should be noted that method (17) can be unstable for the values $k > 2$. But when obtaining a one-step form, it will be stable.

4. The construction of some simple methods and their application to the calculation of definite integrals

First let us consider constructing stable methods of type (5). As is known, for the case $s = k$ we can construct stable methods with the degree $p \le 8$ having the explicit and implicit type for the $k \le 6$. Therefore, let us consider the case when $s < k$ and put $k = 2$. In this case the integer m can take the value $m = 1$ and the maximal value of the degree for this method will be equal to $p = 3$, which can be written as:

$$y_{n+1} = y_n + h(5f_n + 8f_{n+1} - f_{n+2})/12. \tag{18}$$

And now let us put $k = 3$ and $m = 2$, then we get:

$$y_{n+1} = y_n + h(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3})/24. \tag{19}$$

But for the case $k = 4$ and $m = 3$, the stable method can be written as:

$$\begin{aligned} y_{n+1} &= y_n + h(251f_n + 646f_{n+1} - \\ &- 264f_{n+2} + 106f_{n+3} - 19f_{n+4})/720. \end{aligned} \tag{20}$$

Method (19) has the degree $p = 4$ but method (20) has the degree $p = 5$. In [11] it is proved that for the coefficients $|\beta_{k-m+i}| > |\beta_{k-m+i+1}|$ and $\beta_{k-m} > 0$ (for $\alpha_{k-m} > 0$), $\beta_{k-m+i} \cdot \beta_{k-m+i+1} < 0$ hold if the coefficients satisfy the condition: $\beta_{k-m+i} \ne 0, \beta_{k-m+i+1} \ne 0$. The above-described methods satisfy these conditions.

Methods (18) and (20) are very simple to use for the calculation of integral (1) so that y_0 is known. Now let us consider the case $k = 3$ and $m = 1$. In this case, the method with the maximal degree can be written as:

$$\begin{aligned} y_{n+2} &= (8y_{n+1} + 11y_n)/19 + \\ &+ h(10f_n + 57f_{n+1} + 24f_{n+2} - f_{n+3})/57. \end{aligned} \tag{21}$$

This method is stable and has the degree $p = 5$. If we replace in this method the step size h with $h/2$, then we get:

$$\begin{aligned} y_{n+1} &= (8y_{n+1/2} + 11y_n)/19 + \\ &+ h(10f_n + 57f_{n+1/2} + 24f_{n+1} - f_{n+3/2})/114. \end{aligned} \tag{22}$$

Using of method (22) is more difficult than method (20) so that the term of $y_{n+1/2}$ participates in method (22). To find this value we can use method (19) or method (20). But if we consider the case $m = 0$ and $k = 2$, then we get the following stable method with the degree $p = 4$:

$$y_{n+2} = y_n + h(f_{n+2} + 4f_{n+1} + f_n)/3,$$

and in this method after replacing of h by the $h/2$, we get:

$$y_{n+1} = y_n + h(f_{n+1} + 4f_{n+1/2} + f_n)/6, \tag{23}$$

which can be applied to calculation of the integral (1) in separate form.

Now let us consider the construction of stable method of type (8) and put $k = 1$. In this case from the method (8), we can obtain stable methods having the different order of accuracy. For instance, the method with the degree $p = 6$ can be written as:

$$\begin{aligned} y_{n+1} &= y_n + h(f_n + f_{n+1})/12 + \\ &+ 5h(f_{n+\alpha} + f_{n+1-\alpha})/12, \alpha = 1/2 - \sqrt{5}/10. \end{aligned} \tag{24}$$

This method is implicit. In this case, the explicit method with the degree $p = 5$ can be written as:

$$y_{n+1} = y_n + hf_n / 9 + h((16+\sqrt{6}) f_{n+3/5-\alpha} + \\ + (16-\sqrt{6}) f_{n+3/5+\alpha}) / 36, \alpha = \frac{\sqrt{6}}{10}. \quad (25)$$

If we suppose that $\beta_i = 0 (i = 0,1,...,k)$, then it can be proved that there are stable methods with the degree $p = 2k+2$. In the case $k=1$ the method with the degree $p=4$ has the following form:

$$y_{n+1} = y_n + h(f_{n+l} + f_{n+1-l})/2, l = 1/2 - \sqrt{3}/6. \quad (26)$$

And now let us consider the construction of the multistep second derivative methods. As was noted above, if these methods are stable then the degree of these methods satisfies the condition $p \le 2k+2$. First, let us put $s = k = l$ and $k=1$. Then from method (18) we can obtain the following methods [27-30]:

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/2 + h^2(-f'_{n+1} + f'_n)/12, \\ R_n = h^5 f^{(IV)}_{(\xi)} / 720 \quad (27)$$

$$y_{n+1} = y_n + h(f_{n+1} + 2f_n)/2 + h^2 f'_n / 6, \\ R_n = -h^4 f^{(3)}_{(\xi)} / 72, \quad (28)$$

$$y_{n+1} = y_n + hf_n + h^2 f'_n / 2, \\ R_n = h^3 f''_{(\xi)} / 6. \quad (29)$$

Note that method of (27) is implicit, method (29) is explicit, but method (28) depending on its application can be explicit or implicit. In our case method (28) is explicit because function $f(x)$ is independent of the function of $y(x)$.

And now let us consider the case $k=2$. In this case from the (14) receive [11, p.288]:

$$y_{n+2} = y_{n+1} + h(-f_{n+1} + 3f_n)/2 \\ + h^2(17 f'_{n+1} + 7 f'_n)/12, \quad (30)$$

$$y_{n+2} = y_{n+1} + h(101 f_{n+2} + 128 f_{n+1} + 11 f_n)/240 \\ + h^2(-13 f'_{n+2} + 40 f'_{n+1} + 3 f'_n)/240. \quad (31)$$

These methods are stable and has the degree $p=4$ and $p=6$, respectively.

Now let us consider the construction of stable stepwise second derivative methods. It is clear that if $m=1$ then $k \ge 2$. Therefore, consider the case $m=1$ and $k=2$. In this case the stable method with the degree $p=6$ can be written as [11, p.288]:

$$y_{n+1} = y_n + h(11 f_{n+2} + 128 f_{n+1} + 101 f_n)/240 \\ + h^2(-3 f'_{n+2} - 40 f'_{n+1} + 3 f'_n)/240. \quad (32)$$

For the construction of more exact methods, let us put $k=3$ and $m=1$. In this case the stable method with the degree $p=9$ can be written as follows:

$$y_{n+2} = (416 y_{n+1} - 103 y_n)/313 + h(157 f_{n+3} + 1123 f_{n+2} \\ + 8451 f_{n+1} - 2830 f_n)/25353 + h^2(-11 f'_{n+3} - 630 f'_{n+2} \\ + 1557 f'_{n+1} - 92 f'_n)/8451. \quad (33)$$

If in the case $k=3$, put $m=2$, then we get the following one-step method with the degree $p=9$ [11, p.289]:

$$y_{n+1} = y_n + h(1985 f_{n+3} + 12015 f_{n+2} + 42255 f_{n+1} + \\ + 34465 f_n)/90720 + h^2(-163 f'_{n+3} - 2421 f'_{n+2} - \\ - 7659 f'_{n+1} + 1283 f'_n)/30240. \quad (34)$$

Stable methods (32)-(34) have maximal degrees.

As noted above, when using the highest derivatives of solving the problems under consideration in constructing numerical methods, the degree values for these methods can be greater.

For instance, the following methods:

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/2 - h^2(f'_{n+1} - f'_n)/10 + h^3(f''_{n+1} - f''_n)/120, \tag{35}$$

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/2 - 3h^2(f'_{n+1} - f'_n)/28 + h^3(f''_{n+1} - f''_n)/84 - h^4(f'''_{n+1} - f'''_n)/1680. \tag{36}$$

Methods (35) and (36) are stable and have the degree $p = 6$ and $p = 8$, respectively. Please note that the methods described above do not allow some difficulties in their application to the calculation of definite integrals.

For the illustration of the obtained results, let us apply methods (18)-(20) and methods (24)-(26) to calculate the following definite integral:

$$I = \lambda \int_0^1 \exp(\lambda s) ds. \tag{37}$$

This problem has been reduced to solving the problem: $y'(x) = \lambda \exp(\lambda x), y(0) = 0$ the exact solution for which can be written as: $y(x) = \exp(\lambda x) - 1$. This problem has been solved by using of stepwise methods (18)-(20), hybrid methods (24)-(26) and multistep multiderivative methods (27), (35) and (36). To compare the obtained results, the problem under consideration was solved for various values of the constant λ and for the value $h = 0{,}1$.

The results of the calculation of given definite integral are summarized in Table 1 and Table 2.

Table 1

Results for solving example in the case $\lambda = \pm 1$

	λ=1	λ=-1
Method 27	2.39E-7	8.78E-8
Method 35	1.70E-11	6.27E-12
Method 36	2.22E-16	3.33E-16
Method 18	7.43E-5	2.54E-5
Method 19	4.91E-6	1.54E-6
Method 20	3.65E-7	1.05E-7
Method 26	3.98E-8	1.46E-8
Method 25	2.38E-10	8.79E-11
Method 24	1.14E-12	4.18E-13

Table 2

Results for solving example in the case $\lambda = \pm 5$

	λ=5	λ=-5
Method 27	0.36624	8.57E-5
Method 35	2.27E-5	1.53E-7
Method 36	2.25E-8	1.52E-10
Method 18	0.926199	4.32E-3
Method 19	0.366244	1.12E-5
Method 20	0.16476	3.21E-4
Method 26	2.11E-3	1.43E-5
Method 25	6.30E-5	4.31E-7
Method 24	1.51E-6	1.02E-8

For construction of the stable methods with high degrees, we propose three methods. And we give some specific methods of stepwise, hybrid, and multistep types of secondary derivatives, which are compared by applying them to solve example (37).

Note.

Let us consider the calculation of double definite integral which can be written as follows:

$$DI = \int_a^b \int_c^d f(s,t)dsdt. \tag{38}$$

Usually, with the calculation of these integrals of type (38) we encounter the finding of the volume of a geometric figure. For the calculation of integral (1), we propose using the following function:

$$u(x,y) = \int_a^x \int_c^y f(s,t)dsdt, \quad a \le x \le b; \quad c \le y \le d. \tag{39}$$

It follows from here that $u(b,d) = \Delta I$.

As was noted above, the aim of our study is the calculation of integral (38) reduced to solving the initial-value problem for the ODEs.

It is not difficult to understand that integral (39) can be written as (see Fichtengolds…):

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = f(x,y); \quad a \le x \le b; \quad c \le y \le d. \tag{40}$$

To obtain the equivalent problem to calculate integral (39) let us write the partial differential equation of hyperbolic type by the following problem:

$$\frac{\partial^2 u}{\partial x \partial y} = f(x,y), \quad u(a,y) = u(x,c) = 0, \quad a \le x \le b; \quad c \le y \le d. \tag{41}$$

This problem can be reduced to the following:

$$\frac{\partial^2 \bar{u}}{\partial s^2} - \frac{\partial^2 \bar{u}}{\partial t^2} = \varphi(s,t), \quad \bar{u}(s,t) = u(s+t, s-t), \tag{42}$$

$$\varphi(s,t) = f(s+t, s-t), \quad s \ge t.$$

As is known, problem (41) can be solved by using the finite-difference method. In simple form we can use the following formula:

$$\left. \left(\frac{\partial^2 u}{\partial x \partial y} \right) \right|_{\substack{x=x_i \\ y=y_j}} = \frac{1}{h\tau} \left(u_{i,j} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1} \right). \tag{43}$$

Here $u_{i,j} = u(x_i, y_j)$.

It is known that the double definite integral can be calculated from the following formula:

$$\int_a^b \int_c^d f(s,t)dsdt = u(b,d) - u(a,d) - u(b,c) + u(a,c). \tag{44}$$

If we replace a,b,c and d with the x_{i-1}, x_i, y_{j-1} and y_j, then from formula (44) follows method (43) for the step-size $h = \tau = 1$. Formula (43) is simple and can be used as the recurrent formula. But for the construction of more exact formula, we need to use methods which cannot be presented as the recurrent formula and in this case to find unknowns $u_{i,j} \quad (i = 0,1,..., n; \ j = 0,1,..., m)$ we get the linear system of algebraic equations. These difficulties arise also in solving problem (42). Thus, by the above description, we get that replacing the calculation of integral (39) with solving problem (41) or (42) is not expedient. By taking into account these disadvantages of the described method, let us reduce the calculation of integral (38) to solving the initial-value problem for the system of first-order ODEs. For this purpose let us denote by

$$F(x, y) = \int_{c}^{y} f(x, t)dt.$$

Then the double integral (39) can be written as:

$$F_y'(x, y) = f(x_i, y), \quad F(x_i, c) = 0, \quad i = 0,1,2,..., n, \tag{45}$$

$$u_x'(x, d) = F(x, d), \quad u(a, d) = 0. \tag{46}$$

By solving these systems, we can find the value $u(b,d)$. The equations of these systems are not separated. Therefore, we can investigate system (45) and system (46) for the fixed value of the variable i. For example, let us consider the following:

$$\sum_{j=0}^{k} \alpha_i F(x_j, y_{n+i}) = h\sum_{i=0}^{k} \beta_i f(x_j, y_{n+i}) + h\sum_{i=0}^{k} \gamma_i f(x_j, y_{n+i+\nu_i}) \quad (j = 0,1,.., n), \tag{47}$$

$(|\nu_i| < 1; i = 0,1,..., k).$

For the simplicity, let us consider the case when $\gamma_i = 0 \quad (i = 0,1,..., k)$. It is evident that this method can be applied to solvethe problem (46) and this case receives:

$$\sum_{i=0}^{k} \alpha_i u(x_{n+i}, d) = h\sum_{i=0}^{k} \beta_i F(x_{n+i}, d).$$

As is known, this is ordinary multistep method with constant coefficients and for its application the initial values $u(x_0,d), u(x_1,d),...u(x_{k-1},d)$ must be known. By taking into account that the value $u(x_0, y) = 0$, is known that one of the simple methods to solve problem (47) are the one-step methods. Of the one-step methods, explicit Euler and trapezoidal methods are most popular. Therefore, let us consider the following trapezoidal method which can be written as follows:

$$u(x_{n+1}, d) = u(x_n, d) + h(F(x_{n+1}, d) + F(x_n, d)) / 2. \tag{48}$$

This method has the degree $p = 2$. To construct more exact methods let us use method (47). In this case we can construct the stable method with the degree $p = 6$ which can be written as follows:

$$u(x_{n+1}, d) = u(x_n, d) + h(F(x_{n+1}, d) + F(x_n, d)) / 12 + 5hF(x_{n+\alpha}, d) + F(x_{n+1-\alpha}, d)) / 12, \tag{49}$$

$\alpha = 1/2 - \sqrt{5}/10.$

As described above, it is shown here that when applying the methodology proposed here, the degree of accuracy can be increased. As was shown by simple methods, the values of the degree of accuracy for modified methods can more than double (to compare methods (48) and (49)). In fact, if we use only hybrid points, we get the following method:

$$u(x_{n+1}, d) = u(x_n, d) + h(F(x_{n+l}, d) + F(x_{n+1-l}, d)) / 2, \quad l = 1/2 - \sqrt{3}/6.$$

This method is stable and has the degree $p = 4$.

Thus, we prove that using some method, the calculation of double definite integrals can be replaced by solving the initial problem for ODEs. And using one of the methods used in solving ODEs, we can calculate the value of the presented double definite integrals. Note that, as described above, the calculation of double integrals can be replaced to solve some problems for second-order hybrid differential equations. It is known that to solve this problem, specialists used several numerical methods, which resulted in a system of algebraic or ordinary differential equations. In the second case, the need arises for solving ODEs. Using a wide arsenal of existing numerical methods, we can solve the obtained problem. To build methods with high degrees, as noted above, it is necessary to use hybrid methods. Note that for this purpose it is possible to use multistep multi-derivative methods, the application of which for solving the initial value problem for first-order ODEs has been fundamentally investigated.

Note that we have shown here how to reduce the calculation of double integrals to solving the initial problem for ODEs. But it does not follow that these problems are equivalent.

Note that the calculation of the double integral can be reduced to solving the following initial-value problems:

$$F'(x_i, y) = f(x_i, y), F(x_i, c) = 0 (i = 0,1,\ldots,n),$$
$$u'(x,d) = F(x,d), u(a,d) = 0,$$

here $F(x, y) = \int_c^y f(x,t)dt.$

5. Conclusion

As noted above, some methods for calculating a definite integral and their application for calculating definite integrals are described here. Some comparison of the methods recommended for calculating definite integrals is given. It is shown that the behavior of errors when using these methods depends on its type. For this purpose, the results were used, which were summarized in Tables 1 and 2 and obtained for the step size $h = 0,1$. It is also shown that errors mainly depend on the properties of the solution to the problems under consideration. As can be seen from Table 2, the results are obtained by stepwise methods for $h = 0,1$ and $m = 5$ and are not satisfied with the results, which should be considered normal. But the results obtained for the step size $h = 0,1$ can be considered normal. We hope that the method described above will find its wide application in solving scientific and engineering problems. To demonstrate the advantages of the method described here, the application of the aforementioned method for calculating double definite integrals is considered. And to calculate the values of specific double integrals, the methods proposed here are applied. Taking into account the results obtained here, it was found that both methods are of both theoretical and practical interest. This method is a new direction in the calculation of definite integrals, and we believe that it will find his followers.

This paper is an extended version of the report presented by Imanova M.N. at the International Conference on Innovative Applied Energy (IAPE'19).

Acknowledgment

The author expresses her gratitude to Academician Telman Aliev for the suggestion to investigate the computational aspects of our problem. This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan – Grant No EIF-KETPL-2-20151(25)-56/07/1.

References

[1] G.M. Fichtengolts, Course of differential and integral calculus, Nauka, 1970.
[2] P. Michael, Fitz Analog Communication Theory, The Ohio State University, 2001.
[3] T.A. Aliev, N. E. Rzaeva, Techniques for Determining Robust Estimates of Correlation Functions for Random Noisy Signals. Measurement Techniques. 60 No.4 (2017) 343–349.
[4] I.S. Berezin, N.P. Jidkov, Computational methods (Russian), Moscow, Nauka, 1 (1966).
[5] A. Quarteroni, R. Saccar, F. Saleri, Numerical Mathematics, Springer, 2007.

[6] L. Euler, Integral calculus, Moscow, Gostehus-Date, II (1957).
[7] A.N. Krylov, Lectures on approximate calculations, Moscow, Gocteh-izdat, 1950, 400 p.
[8] G. Dahlquist, Convergence and stability in the numerical integration of ordinary differential equations, Math.Scand. No.4 (1956) 33-53.
[9] P. Henrici, Discrete variable methods in ordinary differential equation, Wiley, New York, 1962.
[10] R.J. Lambert, Numerical Methods for ordinary Differential Systems: the initial value problem, John Wiley and Sons Ltd, Chichester.
[11] G. Mehdiyeva, V. Ibrahimov, On the research of multistep methods with constant coefficients, LAP LAMBERT Academic Publishing, 2013, 314.
[12] H. Brunner, Marginal stability and stabilization in the numerical integration of ordinary differential equations, Math. of Coraput. No.111 (1970) 635- 646.
[13] V.R. Ibrahimov, One non-linear method for the numerical solution of the Cauchy problem for ordinary differential equations, Dif. Equations, Bulgarian. (1982) 310-319.
[14] V.R. Ibrahimov, On some properties of Richardson extrapolation, Dif. Equations, Russian. 26 No.12 (1990) 2170-2173.
[15] Jack Williams, Frank de Hoog, A Class of A- Stable Advanced Multistep Methods Mathematics of computation. 28 No.125 (1974) 163-177.
[16] V.R. Ibrahimov, On a relation between order and degree for stable forward jumping formula, Zh Vychis Mat. No. 7 (1990) 1045-1056.
[17] J.C. Butcher, Numerical Methods for Ordinary differential equation, The University of Auckland, New Zeland, 2008, 463 p.
[18] C.W. Gear, Hybrid methods for initial value problems in ordinary differential equations, SIAM, J. Numer. Anal. 2 (1965) 69-86.
[19] G.K. Gupta, A polynomial representation of hybrid methods for solving ordinary differential equations, Mathematics of comp. 33 No.148 (1979) 1251-1256.
[20] E.A. Areo, R.A. Ademiluyi, P.O. Babatola, Accurate collocation multistep method for integration of first order ordinary differential equations, J.of Modern Math.and Statistics. 1-6 (2008).
[21] G. Mehdiyeva, V. Ibrahimov, M. Imanova, On a way for constructing numerical methods on the joint of multistep and hybrid methods, World Academy of Science, engineering and Technology, Paris. (2011) 240-243.
[22] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, On one application of forward jumping methods, Applied Numerical Mathematics. 72 (2013) 234–245.
[23] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Some refinement of the notion of symmetry for the Volterra integral equations and the construction of symmetrical methods to solve them, Journal of Computational and Applied Mathematics. 306 (2016) 1–9.
[24] A. Makroglou, Hybrid methods in the numerical solution of Volterra integro-differential equations, Journal of Numerical Analysis. No.2 (1982) 21-35.
[25] V.I. Krylov, V.V. Bobkov, P.I. Monastyrniy, Origin of theory or calculation methods, differential equations, Nauka i technika, Minsk, 1982.
[26] M. Urabe, An implicit one-step method of high-order accuracy for the numerical integration of ordinary differential equations, Numer.Math., No.2 (1970) 151-164.
[27] R.R. Mirzoyev, G.Yu. Mehdiyeva, V.R. Ibrahimov, On an application of a multistep method for solving Volterra integral equations of the second kind, Proceeding the 2010 International Conference on Theoretical and Mathematical Foundations of Computer Science, Orlando, USA, 2010, 46-50.
[28] G. Dahlquist, Stability and error bounds in the numerical integration of ordinary differential equation, Trans. Of the Royal Inst. Of Techn., Stockholm, Sweden, Nr. 1959, 130.
[29] A. Iserles, S.P. Norsett, Two-step methods and Bi-orthogonality, Math. of Comput. No.180 (1987) 543-552.
[30] J. Kobza, Second derivative methods of Adams type, Applikace Mathematicky. No.20 (1975) 389-405.
[31] E. Hairer, S.P. Norsett, G. Wanner, Solving ordinary differential equations. (Russian) Moscow, Mir, 1990.
[32] V. Ibrahimov, On the maximal degree of the k-step Obrechkoff's method, Bulletin of Iranian Mathematical Society. 28 No.1 (2002) 1-28.

Reviewer: Associate Professor A.B. Pashayev, PhD (Mathematics)