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Necessary and sufficient optimality conditions in one discrete non-local optimal control problem

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ABSTRACT

One discrete optimal process control problem described by difference equations with non-local boundary conditions is considered. A necessary and sufficient optimality condition in the form of the discrete maximum principle is proved. In the case of a non-linear convex quality functional, a sufficient optimality condition is proved.

1. Introduction and problem statement

In [1-4] and others, various aspects of Cauchy-type optimal control problems described by ordinary differential, as well as difference equations with initial conditions were studied. A number of necessary first-order optimality conditions are established. In this paper, we study one discrete optimal control problem with non-separated, non-local boundary conditions. Under the assumption of linearity with respect to the phase vector of the problem under investigation, the necessary and sufficient optimality condition in the form of the discrete maximum principle is proved. For this purpose, one modified version of the increment method is used.

Let us consider a discrete controlled process described by a system of linear heterogeneous difference equations

$$x(t+1) = A(t)x(t) + f\big(t, u(t)\big), \qquad t = t_0, t_0+1, \dots, t_1-1 \tag{1}$$

with the boundary conditions

$$L_0 x(t_0) + L_1 x(t_1) = \ell. \tag{2}$$

Here, $L_0,\ L_1$ are prescribed $(n \times n)$ matrices, $A(t)$ is a prescribed $(n \times n)$ discrete matrix function, $f(t,u)$ is a prescribed n-dimensional vector function that is continuous in the set of variables, ℓ is a prescribed constant vector, $t_0,\ t_1$ are prescribed natural numbers, $x(t)$ is a n-dimensional state vector (trajectory), and $u(t)$ is a r-dimensional discrete vector of control actions with values from the prescribed nonempty and bounded set U, i.e.

$$u(t) \in U \subset R^r, \qquad t \in T = \{t_0, t_0+1, \dots, t_1-1\ \}. \tag{3}$$

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We shall call such control functions admissible controls.

The task is to find the minimum value of the functional

$$J(u) = c'x(t_0) + d'x(t_1), \tag{4}$$

determined on the solutions of boundary value problem (1)-(2) generated by all possible admissible controls.

The admissible control that satisfies the minimum value of functional (4) with constraints (1)-(3) shall be called an optimal control, and the corresponding process $(u(t), x(t))$ – an optimal process. Here, $c,\ d$ are prescribed constant vectors of the corresponding dimensionality.

2. The necessary and sufficient optimality condition

Suppose $(u(t), x(t))$ is a fixed, and $(\bar{u}(t) = u(t) + \Delta u(t),\ \bar{x}(t) = x(t) + \Delta x(t))$ – an arbitrary admissible processes.

Then it is clear that the increment $\Delta x(t)$ of the $x(t)$ will be the solution to the problem

$$\Delta x(t+1) = A(t)\Delta x(t) + f\big(t, \bar{u}(t)\big) - f\big(t, u(t)\big), \tag{5}$$

$$L_0 \Delta x(t_0) + L_1 \Delta x(t_1) = 0. \tag{6}$$

Suppose $\psi(t)$, λ are arbitrary n-dimensional vector function and vector, respectively.

From boundary value problem (5)-(6), we get

$$\sum_{t=t_0}^{t_1-1} \psi'(t)\,\Delta x(t+1) = \sum_{t=t_0}^{t_1-1} \psi'(t)\,A(t)\Delta x(t) + \\ + \sum_{t=t_0}^{t_1-1} \psi'(t)\Big(f\big(t, \bar{u}(t)\big) - f\big(t, u(t)\big)\Big), \tag{7}$$

$$\lambda' L_0 \Delta x(t_0) + \lambda' L_1 \Delta x(t_1) = 0. \tag{8}$$

It is easy to show that

$$\sum_{t=t_0}^{t_1-1} \psi'(t)\,\Delta x(t+1) = \psi'(t_1-1)\Delta x(t_1) - \psi'(t_0-1)\Delta x(t_0) + \sum_{t=t_0}^{t_1-1} \psi'(t-1)\,\Delta x(t). \tag{9}$$

Let us introduce the Hamilton-Pontryagin function in the form:

$$H(t, u, \psi) = \psi' \cdot f(t, u)\,.$$

Taking into account identities (7)-(9), the increment of quality functional (4) can be written in the form:

$$\Delta J(u) = J(\bar{u}) - J(u) = \psi'(t_1-1)\Delta x(t_1) - \psi'(t_0-1)\Delta x(t_0) + \sum_{t=t_0}^{t_1-1} \psi'(t-1)\,\Delta x(t) - \\ - \sum_{t=t_0}^{t_1-1} \psi'(t)\,A(t)\Delta x(t) - \sum_{t=t_0}^{t_1-1} \Big(H\big(t, \bar{u}(t), \psi(t)\big) - H\big(t, u(t), \psi(t)\big)\Big) + \\ + c'\Delta x(t_0) + d'\Delta x(t_1) + \lambda' L_0 \Delta x(t_0) + \lambda' L_1 \Delta x(t_1) \tag{10}$$

Hence, grouping similar members, we will have

$$\Delta J(u) = -(\psi(t_0-1) + L_0'\lambda)'\Delta x(t_0) + (\psi(t_1-1) + L_1'\lambda)'\Delta x(t_1) +$$

$$+\sum_{t=t_0}^{t_1-1}\big(\psi(t-1)-A'(t)\,\psi(t)\big)'\,\Delta x(t)+c'\Delta x(t_0)+d'\Delta x(t_1)-$$

$$-\sum_{t=t_0}^{t_1-1}\Big(H\big(t,\bar{u}(t),\psi(t)\big)-H\big(t,u(t),\psi(t)\big)\Big)=(-\psi(t_0-1)+L_0'\lambda+c)'\Delta x(t_0)+$$

$$+(\psi(t_1-1)+L_1'\lambda+d)'\Delta x(t_1)+\sum_{t=t_0}^{t_1-1}\big(\psi(t-1)-A'(t)\;\psi(t)\big)'\,\Delta x(t)- \tag{11}$$

$$-\sum_{t=t_0}^{t_1-1}\Big(H\big(t,\bar{u}(t),\psi(t)\big)-H\big(t,u(t),\psi(t)\big)\Big).$$

Assuming that $\psi(t)$ satisfies the relations

$$\psi(t-1)=A'(t)\;\psi(t), \tag{12}$$

$$\begin{aligned}&\psi(t_0-1)-L_0'\lambda-c=0\,,\\&\psi(t_1-1)+L_1'\lambda+d=0\,,\end{aligned} \tag{13}$$

then the formula of increment (11) takes the form:

$$\Delta J(u)=-\sum_{t=t_0}^{t_1-1}\Big(H\big(t,\bar{u}(t),\psi(t)\big)-H\big(t,u(t),\psi(t)\big)\Big)\,. \tag{14}$$

Using the formula of increment (14), we prove

Theorem 1. Under the assumptions made, it is necessary and sufficient for the optimality of the admissible control $u(t)$ that the inequality

$$\sum_{t=t_0}^{t_1-1}\Big(H\big(t,v(t),\psi(t)\big)-H\big(t,u(t),\psi(t)\big)\Big)\le 0 \tag{15}$$

holds for all $v(t)\in U,\, t\in T$.

Proof. Necessity. Suppose the control $u(t)$ is optimal. Then it follows from (14) that

$$\sum_{t=t_0}^{t_1-1}\Big(H\big(t,\bar{u}(t),\psi(t)\big)-H\big(t,u(t),\psi(t)\big)\Big)\le 0\,. \tag{16}$$

Assuming

$$\Delta u(t)=v(t)-u(t)\,,\quad t\in T\,,$$

where $v(t)$ arbitrary admissible control, from (16) we get that inequality (15) holds.

This proves the necessary part of the theorem.

Sufficiency. Suppose condition (15) is satisfied. We shall prove that it is an optimal control. It follows from condition (15) that for any $v(t)\in U,\, t\in T$

$$J(v)-J(u)=\sum_{t=t_0}^{t_1-1}\Big(H\big(t,v(t),\psi(t)\big)-H\big(t,u(t),\psi(t)\big)\Big)\ge 0\,.$$

This proves the theorem.

3. Sufficient optimality condition

Suppose we need to find the minimum value of the functional

$$J(u) = \varphi\big(x(t_0), x(t_1)\big) \tag{17}$$

under constraints (1)-(3) (problem (1)-(3), (17)).

Here, $\varphi(\cdot)$ is a prescribed continuously differentiable convex function.

Using the Taylor formula, let us write the increment of quality functional (17) corresponding to the admissible controls $u(t)$ and $\bar{u}(t) = u(t) + \Delta u(t)$.

We have

$$\Delta J(u) = \frac{\partial \varphi'\big(x(t_0), x(t_1)\big)}{\partial x(t_0)} \Delta x(t_0) + \frac{\partial \varphi'\big(x(t_0), x(t_1)\big)}{\partial x(t_1)} \Delta x(t_1) +$$

$$+ o_1(\|\Delta x(t_0)\| + \|\Delta x(t_1)\|). \tag{18}$$

Suppose $p(t)$ is a n-dimensional vector function that is the solution to the difference equation

$$p(t-1) = A'(t)\, p(t), \tag{19}$$

with the boundary conditions

$$p(t_0 - 1) - L_0' \lambda - \frac{\partial \varphi\big(x(t_0), x(t_1)\big)}{\partial x(t_0)} = 0, \tag{20}$$

$$p(t_1 - 1) + L_1' \lambda + \frac{\partial \varphi\big(x(t_0), x(t_1)\big)}{\partial x(t_1)} = 0\,. \tag{21}$$

From relations (20)-(21), we can exclude the parameter λ. Let us multiply both sides of relation (20) on the left by L_1, and relation (21) by L_0 and add the obtained relations together. Then boundary conditions (20)-(21) will take the form:

$$L_1 p\ \ (t_0 - 1) - L_1 \frac{\partial \varphi\big(x(t_0), x(t_1)\big)}{\partial x(t_0)} + L_0\, p(t_1 - 1) - L_0 \frac{\partial \varphi\big(x(t_0), x(t_1)\big)}{\partial x(t_1)} = 0\,.$$

Let us introduce the Hamilton-Pontryagin function in the form

$$M(t, u, p) = p'(t)\, f(t, u)\,.$$

Taking into account the introduced notation and adjoint system (19)-(21), from the formula of increment (18) we get

$$\Delta J(u) = -\sum_{t=t_0}^{t_1 - 1} \Big(M\big(t, \bar{u}(t), p(t)\big) - M\big(t, u(t), p(t)\big) \Big) + o_1(\|\Delta x(t_0)\| + \|\Delta x(t_1)\|)\,.$$

Since the function $\varphi(x_1, x_2)$ is assumed to be convex, then $o_1(\|\Delta x(t_0)\| + \|\Delta x(t_1)\|) \geq 0$.
Therefore

$$\Delta J(u) \geq -\sum_{t=t_0}^{t_1 - 1} \Big(M\big(t, \bar{u}(t), p(t)\big) - M\big(t, u(t), p(t)\big) \Big)\,. \tag{22}$$

From inequality (22) follows

Theorem 2. If the function $\varphi(x_1, x_2)$ is a convex differentiable function, then it is sufficient for the optimality of the admissible control $u(t)$ in problem (1)-(3), (17) that the maximum condition

$$\sum_{t=t_0}^{t_1-1}\Big(M\big(t, v(t), p(t)\big) - M\big(t, u(t), p(t)\big)\Big) \leq 0$$

holds for all $v(t) \in U$, $t \in T$.

4. Conclusion

In this work, we study one discrete optimal control problem described by an ordinary linear difference equation with a linear non-local boundary condition. A necessary and sufficient optimality condition is proved by the increment method. A similar result is established in the case of a nonlinear but convex quality functional. The obtained result allows solving the problem completely.

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