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Multipoint necessary optimality condition for singular controls in stochastic systems

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A B S T R A C T

The optimal control problem is considered in which the state of the process is determined by the system of Ito stochastic differential equations. The so-called multipoint [1-4] necessary optimality conditions for singular controls in the sense of Pontryagin's maximum principle are established.

1. Introduction

Stochastic differential equations are widely used in physics, chemistry, biology, etc. [5-8]. Therefore, the study of optimal control problems described by stochastic differential equations is relevant.

There is a wide range of literature devoted to various aspects of optimal control problems for processes described by Ito stochastic systems [9-13] and others. In these papers, the strongest necessary first-order optimality condition is obtained in the form of Pontryagin's stochastic maximum principle, which in some cases gives an explicit form of optimal control. A special case was also studied in [13], i.e. the case of degeneration of a stochastic analogue of Pontryagin's maximum principle [14], and a number of necessary optimality conditions for special controls are introduced.

This paper also investigates a special case. It was the first to establish the multipoint necessary optimality condition for singular, in the sense of Pontryagin's maximum principle, controls. Note that in this paper, when studying a special case, we use the stochastic analogue of the method proposed for the deterministic case [2, 3, 15], etc.

2. Problem statement and auxiliary facts

Suppose $(\Omega, \mathrm{F}, \mathrm{P})$ is a full probabilistic space with an allocated non-decreasing stream of σ –

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algebras F^t, where $F^t = \sigma(w(s),\ t_0 \le s \le t)$, and $w(t)$ is a n-dimensional standard Wiener process. $L_F^2(t_0, t_1;\ R^n)$ is the space of concerted processes measurable in (t, ω) and F^t, $x(t,\omega): [t_0, t_1]: \Omega \to R^n$, for which

$$E \int_{t_0}^{t_1} \|x(t)\|^2 \, dt < +\infty .$$

Hereinafter, E is the symbol of mathematical expectation.

Let us consider the following nonlinear stochastic differential equation

$$dx(t) = f\big(t, x(t), u(t)\big)dt + \sigma\big(t, x(t)\big)\, dw(t), \qquad t \in (t_0, t_1]\,, \tag{1}$$

with the initial condition

$$x(t_0) = x_0. \tag{2}$$

Here, $x(t) \in L_F^2(t_0, t_1;\ R^n)$ is a state vector; $f(t, x, u)$ is a prescribed n-dimensional vector function continuous in a set of variables together with up to second-order partial derivatives with respect to x; $\sigma(t, x)$: $T \times R^n \to R^{n\times n}$ is a $(n \times n)$ matrix function continuous in a set of variables together with up to second-order partial derivatives with respect to x.

$$u(t,\omega) \in U_d \equiv \{u(.,.) \in L_F^2(t_0, t_1;\ R^r) \mid u(.,.) \in U \subset R^r\}, \tag{3}$$

where U is a given nonempty, bounded set. Let us call U_d the set of admissible controls.

From now on, it is assumed everywhere that a unique solution $x(t)$ of system (1)-(2) with continuous trajectories corresponds to every admissible control $u(t) \in L_F^2(t_0, t_1;\ R^r)$.

The objective is to find the minimum value of the terminal functional

$$I(u) = E\big\{\varphi\big(x(t_1)\big)\big\}\,. \tag{4}$$

Here, $\varphi(x)$ is a given twice continuously differentiable scalar function.

Let us call the admissible control $u(t)$, which is a solution to the problem of the minimum of functional (4) under constraints (1)-(3) (problem (1)-(4)) the optimal control, and the corresponding process $\big(u(t), x(t)\big)$ – the optimal process.

Suppose $\big(u(t), x(t)\big)$ is an optimal process, and $\big(\bar{u}(t) = u(t) + \Delta u(t), \bar{x}(t) = x(t) + \Delta x(t)\big)$ is some admissible process in problem (1)-(4).

Let us introduce a stochastic analog of the Hamilton-Pontryagin function:

$$H(t, x, u, \psi) = \psi'\, f(t, x, u),$$

where $\psi(t) \in L_F^2(t_0, t_1;\ R^n)$ is a random process whose stochastic differential has the form:

$$d\,\psi(t) = \alpha(t)\, dt + \beta(t) dw(t).$$

Here, $\alpha(t)$ is a n-dimensional measurable and bounded vector function, $\beta(t) \in L_F^2(t_0, t_1;\ R^{n\times n})$.

Further, for simplicity, we introduce the notation of type:

$$H_x[t] = H_x\big(t, x(t), u(t), \psi(t)\big),\ \ H_{xx}[t] = H_{xx}\big(t, x(t), u(t), \psi(t)\big),$$

$$f_x[t] = f_x\big(t, x(t), u(t)\big),\ \ \sigma_x[t] = \sigma_x\big(t, x(t)\big),\ \ \sigma_{xx}[t] = \sigma_{xx}\big(t, x(t)\big),$$

$$\Delta_{\bar{u}(t)} H\big(t, x(t), u(t), \psi(t)\big) = H\big(t, x(t), \bar{u}(t), \psi(t)\big) - H\big(t, x(t), u(t), \psi(t)\big),$$

$$\Delta_{\bar{u}(t)} f\big(t, x(t), u(t)\big) = f\big(t, x(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big).$$

We require that the random processes $\psi(t) \in L_F^2(t_0, t_1;\ R^n)$, $\beta(t) \in L_F^2(t_0, t_1;\ R^{n\times n})$ be a solution of the adjoint system:

$$\begin{cases} d\psi(t) = -(H_x[t] + \beta(t)\,\sigma_x[t])dt + \beta(t)\,dw(t), \\ \psi(t_1) = -\varphi_x\big(x(t_1)\big). \end{cases} \tag{5}$$

Then the increment of functional (4) corresponding to the controls $u(t)$ and $\bar{u}(t)$, can be represented as

$$S(\bar{u}) - S(u) = E\left\{-\int_{t_0}^{t_1} \Delta_{\bar{u}} H[t]\,dt + \frac{1}{2}\,\Delta x'(t_1)\,\varphi_x\big(x(t_1)\big)\,\Delta x(t_1) - \int_{t_0}^{t_1} \Delta_{\bar{u}} H'_x[t]\,\Delta x(t)\,dt - \right.$$
$$\left. -\frac{1}{2}\int_{t_0}^{t_1} \Delta x'(t)\,(H_{xx}[t] + \beta(t)\,\sigma_{xx}[t])\,\Delta x(t)\,dt \right\} + \eta_1\big(\Delta u(t)\big). \tag{6}$$

Here,

$$\eta_1\big(\Delta u(t)\big) = E\left\{o_1(\|\Delta x(t_1)\|^2) - o_2(\|\Delta x(t)\|^2) - \frac{1}{2}\int_{t_0}^{t_1} \Delta x'(t)\Delta_{\bar{u}} H_{xx}\,[t]\Delta x(t)\right\},$$

where $\|\alpha\|$ is the norm of the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)'$ is determined from the formula

$$\|\alpha\| = \sum_{i=1}^{k} |\alpha_i|,$$

and the quantities $o_i(\cdot)$, $i = 1, 2$ are determined from the expansions

$$\varphi\big(\bar{x}(t_1)\big) - \varphi\big(x(t_1)\big) = \varphi'_x\big(x(t_1)\big)\,\Delta x(t_1) + \frac{1}{2}\,\Delta x'(t_1)\,\varphi_{xx}\big(x(t)\big)\,\Delta x(t) + o_1(\|\Delta x(t_1)\|^2),$$

$$H\big(t, \bar{x}(t), \bar{u}(t), \psi(t)\big) - H\big(t, x(t), u(t), \psi(t)\big) = H'_x\big(t, x(t), \bar{u}(t), \psi(t)\big)\,\Delta x(t) +$$
$$+\frac{1}{2}\,\Delta x'(t)\,H_{xx}\big(t, x(t), \bar{u}(t), \psi(t)\big)\,\Delta x(t) + o_2(\|\Delta x(t)\|^2).$$

On the other hand, the conditions imposed on $f(t, x, u)$, $\sigma(t, x)$ imply that the increment $\Delta x(t)$ of the trajectory $x(t)$ satisfies the following stochastic linearized problem:

$$\begin{gathered} d\Delta x(t) = (f'_x[t]\,\Delta x(t) + \Delta_{\bar{u}} f[t])\,dt + \sigma'_x[t]\,\Delta x(t)\,dw(t) + \eta_2(\mathrm{t}), \\ \Delta x(t_0) = 0. \end{gathered} \tag{7}$$

where by definition

$$\eta_2(\mathrm{t}) = \big(\Delta_{\bar{u}} f_x[t]\Delta x(t) + o_3(\|\Delta x(t)\|)dt + o_4(\|\Delta x(t)\|)\big)\,dw(t).$$

Here, the quantities $o_i(\cdot)$, $i = 3, 4$ are determined from the expansions

$$f\big(t, \bar{x}(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big) = f'_x\big(t, x(t), \bar{u}(t)\big)\,\Delta x(t) + o_3(\|\Delta x(t)\|),$$
$$\sigma\big(t, \bar{x}(t)\big) - \sigma\big(t, x(t)\big) = \sigma'_x\big(t, x(t)\big)\,\Delta x(t) + o_4(\|\Delta x(t)\|).$$

Note that based on a stochastic analogue of the Cauchy formula [16], the solution to system (7) can be represented as

$$\Delta x(t) = \int_{t_0}^{t} F(t, \tau)\,\Delta_{\bar{u}} f[\tau]\,d\tau + \eta_3(\mathrm{t}), \tag{8}$$

where

$$\eta_3(t) = \int_{t_0}^{t} F(t,\tau)\,\eta_2(\tau)\,d\tau,$$

and $(n \times n)$ – matrix function, $F(t,\tau)$ is a solution of the linear stochastic matrix differential equation

$$dF(t,\tau) = f_x'[t]\,F(t,\tau)\,dt + \sigma_x'[t]\,F(t,\tau)\,dw(t),$$
$$F(t,t) = I \quad (I - (n \times n) - \text{identity matrix}).$$

Representation (8) allows obtaining the following identities for some terms of formula (6):

$$\Delta x'(t_1)\,\varphi_{xx}\big(x(t_1)\big)\,\Delta x(t_1) = \int_{t_0}^{t_1}\int_{t_0}^{t_1} \Delta_{\bar u} f'[\tau]\,F(t_1,\tau)\,\varphi_{xx}\big(x(t_1)\big)\,F(t_1,s)\,\Delta_{\bar u} f[s]\,ds\,d\tau +$$
$$+\left(\int_{t_0}^{t_1} \Delta_{\bar u} f[\tau]\,F(t_1,\tau)\,d\tau\right)'\varphi_{xx}\big(x(t_1)\big)\,\eta_3(t_1) + \eta_3(t_1)\,\varphi_{xx}\big(x(t_1)\big)\,\Delta x(t_1), \qquad (9)$$

$$\int_{t_0}^{t_1} \Delta_{\bar u} H_x'[t]\,\Delta x(t)\,dt = \int_{t_0}^{t_1}\left[\int_{t}^{t_1} \Delta_{\bar u} H_x'[\tau]\,F(\tau,t)\,d\tau\right]\Delta_{\bar u} f[t]\,dt + \int_{t_0}^{t_1} \Delta_{\bar u} H_x'[t]\,\eta_3(t)\,dt, \qquad (10)$$

$$\int_{t_0}^{t_1} \Delta x'(t)\,\big[H_{xx}[t] + \beta(t)\,\sigma_{xx}[t]\big]\,\Delta x(t)\,dt =$$
$$= \int_{t_0}^{t_1}\int_{t_0}^{t_1} \Delta_{\bar u} f'[\tau]\left[\int_{\max(\tau,s)}^{t_1} F(t,\tau)(\,H_{xx}[t] + \beta(t)\,\sigma_{xx}[t])\,F(t,s)\,dt\right]\Delta_{\bar u} f[s]\,ds\,d\tau + \qquad (11)$$
$$+\int_{t_0}^{t_1}\left(\int_{t_0}^{t_1} \Delta_{\bar u} f[\tau]\,F(t,\tau)d\tau\right)'(\,H_{xx}[t] + \beta(t)\,\sigma_{xx}[t])\,\eta_3(t)\,dt +$$
$$+\int_{t_0}^{t_1} \eta_3(t)\,(\,H_{xx}[t] + \beta(t)\,\sigma_{xx}[t])\,\Delta x(t)\,dt\,.$$

By analogy with [2, 3, 15], we introduce the matrix function

$$K(\tau,s) = -F(t_1,\tau)\,\varphi_{xx}\big(x(t_1)\big)\,F(t_1,s) +$$
$$+\int_{\max(\tau,s)}^{t_1} F(t,\tau)(\,H_{xx}[t] + \beta(t)\,\sigma_{xx}[t])\,F(t,s)\,dt. \qquad (12)$$

Now, taking into account identities (9)-(12) in the second-order increment formula (6) of quality criterion (4), we can write it in the following final form:

$$S(\bar u) - S(u) = E\left\{-\int_{t_0}^{t_1} \Delta_{\bar u} H[t]\,dt + \int_{t_0}^{t_1}\int_{t_0}^{t_1} \Delta_{\bar u} f'[\tau]\,K(\tau,s)\,\Delta_{\bar u} f[s]\,d\tau\,ds +\right.$$
$$\left.+\int_{t_0}^{t_1}\left[\int_{t}^{t_1} \Delta_{\bar u} H_x'[\tau]\,F(\tau,t)\,d\tau\right]\Delta_{\bar u} f[t]\,dt\right\} + \eta\big(\Delta u(t)\big). \qquad (13)$$

Here,

$$\eta\big(\Delta u(t)\big) =$$

$$= \eta_1\big(\Delta u(t)\big) + E\left\{\left(\int_{t_0}^{t_1} \Delta_{\bar{u}} f[\tau]\, F(t_1,\tau)\, d\tau\right)' \varphi_{xx}\big(x(t_1)\big)\, \eta_3(t_1) + \eta_3(t_1)\, \varphi_{xx}\big(x(t_1)\big)\, \Delta x(t_1)\right.$$

$$+ \int_{t_0}^{t_1} \Delta_{\bar{u}} H_x'[t]\, \eta_3(t)\, dt + \int_{t_0}^{t_1} \left(\int_{t_0}^{t_1} \Delta_{\bar{u}} f[\tau]\, F(t,\tau) d\tau\right)' (\, H_{xx}[t] + \beta(t)\, \sigma_{xx}[t])\, \eta_3(t)\, dt +$$

$$\left. + \int_{t_0}^{t_1} \eta_3(t)\, (\, H_{xx}[t] + \beta(t)\, \sigma_{xx}[t])\, \Delta x(t)\, dt \right\}.$$

3. Necessary optimality conditions

We proceed to obtain the necessary optimality conditions. From formula (13) we can obtain

Theorem 1 (a stochastic analog of Pontryagin's maximum principle [9-13**])**. Suppose $u(t)$ is the optimal control in problem (1)-(4). Then the control $u(t)$ satisfies on $[t_0, t_1)$ the condition

$$E\ \Delta_v H[\theta] \leq 0,$$

where $\theta \in [t_0, t_1)$ is an arbitrary Lebesgue point of the control $u(t)$, and $v \in U_d$.

By analogy with **[**13, 14**]**, let us call the admissible control $u(t)$ singular, in the sense of Pontryagin's maximum principle, control, if the following relations hold along the process $\big(u(t), x(t)\big)$

$$\Delta_v H[\theta] = 0, \quad \text{п. н.} \quad v \in U_d \ \text{ and } \ \theta \in [t_0, t_1).$$

It is clear that for singular controls the Hamilton-Pontryagin function $H\big(t, x(t), u(t), \psi(t)\big)$ do not depend on the control parameters and, therefore, the maximum condition degenerates.

We investigate the optimality of singular, in the sense of Pontryagin's maximum principle, controls.

The special increment of the singular optimal control $u(t)$ is determined from the formula

$$\Delta u_\varepsilon(t) = \sum_{i=1}^{m} \delta u(t; \varepsilon, \theta_i, l_i, v_i). \tag{14}$$

Here, m is an arbitrary natural number, $\varepsilon > 0$ is a sufficiently small number, $l_i \geq 0,\ i = \overline{1,m}$ are arbitrary numbers, $v_i \in U,\ \ \theta \in [t_0, t_1),\ i = \overline{1,m}$ are Lebesgue points of control, $u(t)$, $t_0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_m < t_1$, a $\delta u(t, \varepsilon, \theta_i, l_i, v_i)$ is a needle control variation:

$$\delta u(t;\ \varepsilon, \theta_i, l_i, v_i) = \begin{cases} v_i - u(t), & t \in [\theta_i, \theta_i + l_i\, \varepsilon), \\ \quad 0, & t \in T \backslash [\theta_i, \theta_i + l_i\, \varepsilon). \end{cases} \tag{15}$$

Summation of needle variations (15) will be determined according to the pattern [1].

Let us denote by $\Delta x_\varepsilon(t)$ the special increment of the trajectory, $x(t)$, corresponding to increment (14) of the control $u(t)$.

By analogy with [13, 14, 17], using the analog of the Gronwall–Bellman lemma, it is proved that the following estimate is valid on the needle control variation:

$$E\|\Delta x_\varepsilon(t)\| \leq N\ \varepsilon, \qquad N = const > 0. \tag{16}$$

In view of (14), (16), we obtain $\eta\big(\Delta u_\varepsilon(t)\big) = o(\varepsilon^2)$.

Taking into account this fact and relation (14), from (13) after some transformations, the optimality force $u(t)$, we obtain the main result of the work

Theorem 2. Suppose $u(t)$ is the optimal singular, in the sense of Pontryagin's maximum principle, control in problem (1)-(4). Then for any natural m, the inequality

$$E\left\{\sum_{i=1}^{m}\sum_{j=1}^{m} l_i\, l_j \Delta_{v_i} f'[\theta_i]\, K(\theta_i,\theta_j)\, \Delta_{v_j} f[\theta_j] + \right.$$

$$\left. + \sum_{i=1}^{m} l_i\, \Delta_{v_i} H_x'[\theta_i] \left[l_i\, \Delta_{v_i} f[\theta_i] + 2\sum_{j=1}^{i-1} l_j\, F(\theta_i,\theta_j)\, \Delta_{v_j} f[\theta_j] \right] \right\} \le 0, \qquad (17)$$

holds for all $\theta_i \in [t_0, t_1)$ – Lebesgue point of the control$u(t)$ and $l_i \ge 0$, $v_i \in U$, $i = \overline{1,m}$, $(t_0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_m < t_1)$.

Note that the established sequence of necessary optimality conditions for singular controls, each of which, independently of the others, must be satisfied along the process $\big(u(t), x(t)\big)$.. Therefore, condition (17) allows significantly narrowing the number of singular controls suspicious of optimality.

From condition (17), we can easily obtain various necessary optimality conditions more convenient for verification.

When $m = 1$, from the theorem follows

Corollary 1. Suppose $u(t)$ is the optimal singular, in the sense of Pontryagin's maximum principle, control in problem (1)-(4), then for its optimality it is necessary that the inequality

$$A(\theta, v) = E\,\{\Delta_v\, f[\theta]\, K\,(\theta,\theta)\, \Delta_v\, f[\theta] + \Delta_v\, H_x[\theta]\, \Delta_v\, f[\theta]\} \le 0, \qquad (18)$$

holds for all $\theta \in [t_0, t_1)$ and $v \in U$.

Note that optimality condition (18) was obtained in [18].

Corollary 2. If $u(t)$ is the optimal singular, in the sense of Pontryagin's maximum principle, control in problem (1)-(4), then along the process $\big(u(t), x(t)\big)$ the conditions

$$A(\theta_1, v_1) \le 0, \quad A(\theta_2, v_2) \le 0, \qquad (19)$$

$$E\{\Delta_{v_2}\, f[\theta_2]\, K\,(\theta_2,\theta_1)\, \Delta_{v_1}\, f[\theta_1]\} \le \sqrt{A(\theta_1, v_1)\cdot A(\theta_2, v_2)}\,, \qquad (20)$$

holds for all $\theta_1, \theta_2 \in [t_0, t_1)$, $\theta_1 \le \theta_2$, and $\forall\ v_1, v_2 \in U$.

The proof follows from the condition of the quadratic trinomial, which is obtained from formula (17) for $m = 2$.

We now note that optimality criterion (17) also remains valid under degeneration of the optimality conditions (18)-(20).

4. Conclusion

Using a stochastic analog of the increment method, which is an effective method in the theory of necessary conditions for optimal control, the necessary first-order optimality conditions are derived in the form of Pontryagin's maximum principle. Next, we consider the case of degeneration of the maximum condition and obtain the multipoint necessary optimality condition for singular controls. From it, a more easily verifiable multipoint necessary condition is obtained for singular, in the sense of the Pontryagin maximum principle, controls.

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