On an optimal control problem for 3D Bianchi integro-differential equations with nonsmooth coefficients under conditions in the geometric middle of the domain

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ABSTRACT

Necessary and sufficient optimality conditions are obtained in the form of L.S. Pontryagin's maximum principle in the study of an optimal control problem described by a three-dimensional boundary value problem defined in the geometric middle of the domain for one third-order integro-differential 3D Bianchi with L_p-coefficients.

1. Introduction

To date, a number of necessary and sufficient optimality conditions have been obtained for various optimal control problems described by hyperbolic equations, as well as equations of mathematical physics, under various assumptions. The emergence and development of the theory of optimal control led to its full-scale application to practical problems, such as control of controlled objects, optimization of dynamical systems, etc. Many of these optimal control problems are described by hyperbolic equations, solutions of which are the subject of numerous works. Problems of optimal control of systems with distributed parameters have numerous applications.

Pontryagin's maximum principle, which is the fundamental result of the theory of necessary first-order optimality conditions, was originally proved (in the linear case by R.V. Gamkrelidze, in the nonlinear case by V.G. Boltyansky) for the optimal control problem described by ordinary differential controls. Subsequently, studies appeared devoted to the derivation of the necessary optimality conditions in more complex control problems with lumped and distributed parameters. Problems of optimal control described by hyperbolic equations under Goursat conditions originate from the works of A.I. Yegorov [1, p.615]. Subsequently, various aspects of the problems of optimal process control described by Goursat-Darboux systems were investigated in the works of K.T. Ahmedov and S.S. Akhiyev [2, p.12], M.J. Mardanov [3, p. 181], K. B. Mansimov [4, p.954;

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5, p. 37], M.J. Mardanov and T.K. Melikov [6, p. 49], T.K. Melikov [7, p. 48; 8, p. 257; 9, p. 229], F.P. Vasilyev [10, p.219], V.I. Plotnikov and V.I. Sumin [11, p. 74], R.A. Bandaliyev, V.S. Guliyev, I.G. Mamedova and A.B. Sadigova [12, p.12], R.A. Bandaliev, V.S. Guliyev, I.G. Mamedov and Y.I. Rustamov [13, p.305] and others.

In the qualitative theory of optimal processes, correct solvability of the considered Goursat boundary value problem holds a special place. Goursat problems for hyperbolic equations with discontinuous coefficients under nonclassical boundary conditions were studied in the works of S.S. Akhiyev [14, p. 785; 15, p.267] and I.G. Mamedov [16, p. 75; 17, p. 55; 18, p. 42; 19, p. 253]. The boundary value problem for the 3D Bianchi integro-differential equation with nonsmooth coefficients under conditions in the geometric middle of the domain was first studied in the works of I.G. Mamedov and A.J. Abdullayev [20, p. 73; 21, p. 4]. Note that this formulation problem generalizes the Goursat problem for the 3D Bianchi integro-differential equation with nonsmooth coefficients, i.e. the considered boundary-value problem in the particular case coincides with the Goursat problem. It should be especially noted that it is known from the literature that the optimal control problem for the 3D Bianchi integro-differential equation with nonsmooth coefficients under conditions in the geometric middle of the domain has not yet been investigated. It is for this reason, therefore, that this study is devoted to deriving the necessary optimality conditions such as L.S. Pontryagin's maximum principle for one optimal control problem with distributed parameters described by the 3D Bianchi integro-differential equations with L_p-coefficients under conditions in the geometric middle of the domain.

The obtained results can be used in the theory of optimal processes for the propagation of L.S. Pontryagin's maximum principle to various controlled processes described by 3D Bianchi integro-differential equations with discontinuous coefficients (in other words, with L_p-coefficients) in isotropic Sobolev spaces with dominant mixed derivatives. In this paper, under certain assumptions, this is demonstrated for one model case.

2. Problem statement

Suppose the controlled object is described by the 3D Bianchi integro-differential equation:

$$\begin{aligned}
(V_{1,1,1}u)(x,y,z) &\equiv u_{xyz}(x,y,z) + A_{0,0,0}u(x,y,z) + A_{1,0,0}u_x(x,y,z) + \\
&+ A_{0,1,0}u_y(x,y,z) + A_{0,0,1}u_z(x,y,z) + A_{1,1,0}u_{xy}(x,y,z) + A_{0,1,1}u_{yz}(x,y,z) + \\
&+ A_{1,0,1}u_{xz}(x,y,z) + \int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{y_0y_1}}^{y}\int\limits_{\sqrt{z_0z_1}}^{z}\big[K_{0,0,0}(\tau,\xi,\eta;x,y,z)u(\tau,\xi,\eta) + K_{1,0,0}(\tau,\xi,\eta;x,y,z)\times \\
&\times u_x(\tau,\xi,\eta) + K_{0,1,0}(\tau,\xi,\eta;x,y,z)u_y(\tau,\xi,\eta) + K_{0,0,1}(\tau,\xi,\eta;x,y,z)\times \\
&\times u_z(\tau,\xi,\eta) + K_{1,1,0}(\tau,\xi,\eta;x,y,z)u_{xy}(\tau,\xi,\eta + K_{0,1,1}(\tau,\xi,\eta;x,y,z)\times \\
&\times u_{yz}(\tau,\xi,\eta) + K_{1,0,1}(\tau,\xi,\eta;x,y,z)u_{xz}(\tau,\xi,\eta)\big]d\tau d\xi d\eta = \varphi(x,y,z,v(x,y,z)), \\
&(x,y,z) \in G = G_1 \times G_2 \times G_3
\end{aligned} \tag{1}$$

under the following conditions of the geometric middle of the domain

$$\begin{cases} V_{0,0,0}u \equiv u(\sqrt{x_0x_1}, \sqrt{y_0y_1}, \sqrt{z_0z_1}) = \varphi_{0,0,0} \\ (V_{1,0,0}u)(x) \equiv u_x(x, \sqrt{y_0y_1}, \sqrt{z_0z_1}) = \varphi_{1,0,0}(x), \\ (V_{0,1,0}u)(y) \equiv u_y(\sqrt{x_0x_1}, y, \sqrt{z_0z_1}) = \varphi_{0,1,0}(y), \\ (V_{0,0,1}u)(z) \equiv u_z(\sqrt{x_0x_1}, \sqrt{y_0y_1}, z) = \varphi_{0,0,1}(z), \\ (V_{1,1,0}u)(x,y) \equiv u_{xy}(x, y, \sqrt{z_0z_1}) = \varphi_{1,1,0}(x,y), \\ (V_{0,1,1}u)(y,z) \equiv u_{yz}(\sqrt{x_0x_1}, y, z) = \varphi_{0,1,1}(y,z), \\ (V_{1,0,1}u)(x,z) \equiv u_{xz}(x, \sqrt{y_0y_1}, z) = \varphi_{1,0,1}(x,z), \end{cases} \tag{2}$$

where $\varphi_{0,0,0} \in R$ is a given number and the rest $\varphi_{i,j,k}$ are given functions satisfying the conditions:

$$\varphi_{1,0,0}(x) \in L_p(G_1), \varphi_{0,1,0}(y) \in L_p(G_2), \varphi_{0,0,1}(z) \in L_p(G_3), \varphi_{1,1,0}(x,y) \in L_p(G_1 \times G_2),$$

$$\varphi_{0,1,1}(y,z) \in L_p(G_2 \times G_3), \varphi_{1,0,1}(x,z) \in L_p(G_1 \times G_3); \quad G_1 = (x_0, x_1), G_2 = (y_0, y_1), G_3 = (z_0, z_1),$$

where $x_0 \geq 0, y_0 \geq 0, z_0 \geq 0$.

It is assumed that the coefficients $A_{i,j,k}(x,y,z)$ and $K_{i,j,k}(\tau,\xi,\eta;x,y,z)$ are nonsmooth functions satisfying only the following conditions:

$$A_{0,0,0}(x,y,z) \in L_p(G),\ A_{1,0,0}(x,y,z) \in L^{x,y,z}_{\infty,p,p}(G), A_{0,1,0}(x,y,z) \in L^{x,y,z}_{p,\infty,p}(G),$$

$$A_{0,0,1}(x,y,z) \in L^{x,y,z}_{p,p,\infty}(G), A_{1,1,0}(x,y,z) \in L^{x,y,z}_{\infty,\infty,p}(G),\ A_{0,1,1}(x,y,z) \in L^{x,y,z}_{p,\infty,\infty}(G),$$

$$A_{1,0,1}(x,y,z) \in L^{x,y,z}_{\infty,p,\infty}(G),\ K_{i,j,k}(\tau,\xi,\eta;x,y,z) \in L_\infty(G \times G).$$

It is also assumed that $\varphi(x,y,z,v)$ is a given function on $G \times R^r$ satisfying the Carathéodory conditions $G \times R^r$ (i.e. $\varphi(x,y,z,v)$ is measurable in (x,y,z) on G for all given $v \in R^r$ and is continuous in v on R^r almost for all given $(x,y,z) \in G$ and for any positive number $\delta > 0$ there exists a function $\varphi^0_\delta(x,y,z) \in L_p(G)$ ыгср that $|\varphi(x,y,z,v(x,y,z))| \leq \varphi^0_\delta(x,y,z)$ almost for all $(x,y,z) \in G$ and all $v \in R^r$ for which $\|v\| = \sum_{i=1}^r |v_i| \leq \delta$; $v(x,y,z) = (v_1(x,y,z), \dots, v_r(x,y,z))$ is the r-dimensional control vector function.

Let the vector function $v(x,y,z) = (v_1(x,y,z), \dots, v_r(x,y,z))$ be measurable and bounded on G and at almost all points $(x,y,z) \in G$ takes its values from some given set $U \subseteq R^r$. Then we will call this vector function an admissible control. The set of all admissible controls is denoted by U_∂.

Now consider the following optimal control problem: find an admissible control$v(x,y,z)$ from U_∂, for which the solution

$$u \in W_p^{(1,1,1)}(G) \equiv \{u \in L_p(G) / D_x^i D_y^j D_z^k u \in L_p(G); i,\ j,k = 0,1\}, 1 \leq p \leq \infty$$

of problem (1)-(2) affords the lowest value to multi-drop functionality

$$S(v) = \sum_{k=1}^{N} [a_k^{(1,0,0)} u(x_k^{(1)}, y_1, z_1) + a_k^{(0,1,0)} u(x_1, y_k^{(1)}, z_1) + a_k^{(0,0,1)} u(x_1, y_1, z_k^{(1)})], \tag{3}$$

where $(x_k^{(1)}, y_k^{(1)}, z_k^{(1)}) \in \bar{G}$ are given points; $a_k^{(1,0,0)}, a_k^{(0,1,0)}, a_k^{(0,0,1)} \in R$ are given numbers. Note that the norm in the Sobolev space $W_p^{(1,1,1)}(G)$ will be defined by the equality $\|u\|_{W_p^{(1,1,1)}(G)} = \sum_{i,j,k=0}^{1} \|D_x^i D_y^j D_z^k u\|_{L_p(G)}$.

It should be especially noted that, in particular, in the case $x_0 = y_0 = z_0 = 0$, conditions (2) coincide with the Goursat conditions for the 3D Bianchi integro-differential equation (1).

3. Formula for incrementing the quality criterion in the integral form and optimality conditions

To obtain the necessary and sufficient optimality conditions, we first find the increments of functional (3). Let $\nu(x,y,z)$ and $\nu(x,y,z)+\Delta\nu(x,y,z)$ be different admissible controls, and $u(x,y,z)$ and $u(x,y,z)+\Delta u(x,y,z)$ are the corresponding solutions to problem (1)-(2) in the space $W_p^{(1,1,1)}(G)$. Then the increment of functional (3) will have the form

$$\Delta S(v)=\sum_{k=1}^{N}[a_k^{(1,0,0)}\Delta u(x_k^{(1)},y_1,z_1)+a_k^{(0,1,0)}\Delta u(x_1,y_k^{(1)},z_1)+a_k^{(0,0,1)}\Delta u(x_1,y_1,z_k^{(1)})]. \tag{4}$$

Obviously, in this case, the vector function $\Delta u\in W_p^{(1,1,1)}(G)$ is a solution of the equation

$$(V_{1,1,1}\Delta u)(x,y,z)=\Delta\varphi(x,y,z), \tag{5}$$

satisfying the trivial conditions of the geometric middle of the domain

$$\begin{aligned}&V_{0,0,0}\Delta u=\Delta u(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1})=0, \qquad V_{1,0,0}\Delta u=\Delta u_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1})=0,\\ &V_{0,1,0}\Delta u=\Delta u_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1})=0,\ \cdots,\ V_{0,1,1}\Delta u=\Delta u_{yz}(\sqrt{x_0x_1},y,z)=0,\\ &V_{1,0,1}\Delta u=\Delta u_{xz}(x,\sqrt{y_0y_1},z)=0,\end{aligned} \tag{6}$$

where

$$\Delta\varphi(x,y,z)=\varphi(x,y,z,v(x,y,z)+\Delta v(x,y,z))-\varphi(x,y,z,v(x,y,z))$$

The operator $V=(V_{0,0,0},V_{1,0,0},V_{0,1,0},V_{0,0,1},V_{1,1,0},V_{0,1,1},V_{1,0,1},V_{1,1,1})$ of problem (1)-(2) acts from $W_p^{(1,1,1)}(G)$ on

$$E_p^{(1,1,1)}\equiv R\times L_p(x_0,x_1)\times L_p(y_0,y_1)\times L_p(z_0,z_1)\times L_p(G_1\times G_2)\times L_p(G_2\times G_3)\times$$
$$\times L_p(G_1\times G_3)\times L_p(G).$$

Note that in the space $E_p^{(1,1,1)}$ we will define the norm in a natural way, using the equality

$$\begin{aligned}\|\varphi\|_{E_p^{(1,1,1)}}=&\|\varphi_{0,0,0}\|_R+\|\varphi_{1,0,0}\|_{L_p(x_0,x_1)}+\|\varphi_{0,1,0}\|_{L_p(y_0,y_1)}+\|\varphi_{0,0,1}\|_{L_p(z_0,z_1)}+\|\varphi_{1,1,0}\|_{L_p(G_1\times G_2)}+\\ &+\|\varphi_{0,1,1}\|_{L_p(G_2\times G_3)}+\|\varphi_{1,0,1}\|_{L_p(G_1\times G_3)}+\|\varphi_{1,1,1}\|_{L_p(G)}.\end{aligned}$$

It is shown that this operator has the adjoint operator

$V^*=(\omega_{0,0,0},\omega_{1,0,0},\omega_{0,1,0},\omega_{0,0,1},\omega_{1,1,0},\omega_{0,1,1},\omega_{1,0,1},\omega_{1,1,1})$ acting in $E_q^{(1,1,1)}$ and satisfying the condition

$$\begin{aligned}f(Vu)=&\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}f_{1,1,1}(x,y,z)(V_{1,1,1}u)(x,y,z)dxdydz+f_{0,0,0}(V_{0,0,0}u)+\int_{\sqrt{x_0x_1}}^{x_1}f_{1,0,0}(x)(V_{1,0,0}u)(x)dx+\\ &+\int_{\sqrt{y_0y_1}}^{y_1}f_{0,1,0}(y)(V_{0,1,0}u)(y)dy+\int_{\sqrt{z_0z_1}}^{z_1}f_{0,0,1}(z)(V_{0,0,1}u)(z)dz+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}f_{1,1,0}(x,y)(V_{1,1,0}u)(x,y)dxdy+\\ &+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}f_{0,1,1}(y,z)(V_{0,1,1}u)(y,z)dydz+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1}f_{1,0,1}(x,z)(V_{1,0,1}u)(x,z)dxdz=\\ &=u(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1})(\omega_{0,0,0}f)+\int_{\sqrt{x_0x_1}}^{x_1}u_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1})(\omega_{1,0,0}f)(x)dx+\\ &+\int_{\sqrt{y_0y_1}}^{y_1}u_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1})(\omega_{0,1,0}f)(y)dy+\int_{\sqrt{z_0z_1}}^{z_1}u_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z)(\omega_{0,0,1}f)(z)dz+\end{aligned}$$

$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}u_{xy}(x,y,\sqrt{z_0z_1})(\omega_{1,1,0}f)(x,y)dxdy+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}u_{yz}(\sqrt{x_0x_1},y,z)(\omega_{0,1,1}f)(y,z)dydz+$$
$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1}u_{xz}(x,\sqrt{y_0y_1},z)(\omega_{1,0,1}f)(x,z)dxdz+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}u_{xyz}(x,y,z)(\omega_{1,1,1}f)(x,y,z)dxdydz\equiv \quad (7)$$
$$\equiv(V^*f)(u),$$

where

$$f=(f_{0,0,0},f_{1,0,0}(x),f_{0,1,0}(y),f_{0,0,1}(z),f_{1,1,0}(x,y),f_{0,1,1}(y,z),f_{1,0,1}(x,z),f_{1,1,1}(x,y,z))\in$$
$$\in E_q^{(1,1,1)}\equiv R\times L_q(x_0,x_1)\times L_q(y_0,y_1)\times L_q(z_0,z_1)\times L_q(G_1\times G_2)\times L_q(G_2\times G_3)\times L_q(G_1\times G_3)\times L_q(G),$$

arbitrary linear bounded functional on $E_p^{(1,1,1)}$, i.e. $\frac{1}{p}+\frac{1}{q}=1$ and u an arbitrary function from $W_p^{(1,1,1)}(G)$.

Now in equality (7), we choose the function $u(x,y,z)\in W_p^{(1,1,1)}(G)$ as a solution to problem (5)-(6), i.e. we put together u the function Δu. Then we get that the equality

$$f(V\Delta u)\equiv\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}f_{1,1,1}(x,y,z)\Delta\varphi(x,y,z)dxdydz=$$
$$=\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}(\omega_{1,1,1}f)(x,y,z)\Delta u_{xyz}(x,y,z)dxdydz\equiv(V^*f)(\Delta u), \quad (8)$$

is true for all $f\in E_q^{(1,1,1)}$. In other words,

$$-\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}f_{1,1,1}(x,y,z)\Delta\varphi(x,y,z)dxdydz+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}(\omega_{1,1,1}f)(x,y,z)\Delta u_{xyz}(x,y,z)dxdydz=0,$$
$$\forall f\in E_q^{(1,1,1)}. \quad (9)$$

The function $\Delta u(x,y,z)$ as an element of the space $W_p^{(1,1,1)}(G)$ satisfies trivial conditions (6). Using the integral representation of functions [22, p. 5] from $W_p^{(1,1,1)}(G)$:

$$u(x,y,z)=u(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1})+\int_{\sqrt{x_0x_1}}^{x}u_\alpha(\alpha,\sqrt{y_0y_1},\sqrt{z_0z_1})d\alpha+$$
$$+\int_{\sqrt{y_0y_1}}^{y}u_\beta(\sqrt{x_0x_1},\beta,\sqrt{z_0z_1})d\beta+\int_{\sqrt{z_0z_1}}^{z}u_\gamma(\sqrt{x_0x_1},\sqrt{y_0y_1},\gamma)d\gamma+$$
$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}u_{\alpha\beta}(\alpha,\beta,\sqrt{z_0z_1})d\alpha d\beta+\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z}u_{\beta\gamma}(\sqrt{x_0x_1},\beta,\gamma)d\beta d\gamma+$$
$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{z_0z_1}}^{z}u_{\alpha\gamma}(\alpha,\sqrt{y_0y_1},\gamma)d\alpha d\gamma+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z}u_{\alpha\beta\gamma}(\alpha,\beta,\gamma)d\alpha d\beta d\gamma$$

we get

$$a_k^{(1,0,0)}\Delta u(x_k^{(1)},y_1,z_1)+a_k^{(0,1,0)}\Delta u(x_1,y_k^{(1)},z_1)+a_k^{(0,0,1)}\Delta u(x_1,y_1,z_k^{(1)})=$$
$$=\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1}R_k(x,y,z)\Delta u_{xyz}(x,y,z)dxdydz$$

where

$$R_k(x,y,z)=a_k^{(1,0,0)}\theta(x_k^{(1)}-x)+a_k^{(0,1,0)}\theta(y_k^{(1)}-y)+a_k^{(0,0,1)}\theta(z_k^{(1)}-z),$$

$$\theta(t)=\begin{cases}1, & t>0\\ 0, & t\le 0\end{cases}$$ is Heaviside function.

Therefore, increment (4) of functional (3) can be represented as

$$\Delta S(v)=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\sum_{k=1}^{N}R_k(x,y,z)\Delta u_{xyz}(x,y,z)dxdydz$$

or

$$\Delta S(v)=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}R(x,y,z)\Delta u_{xyz}(x,y,z)dxdydz, \qquad (10)$$

where

$$R(x,y,z)=\sum_{k=1}^{N}R_k(x,y,z).$$

Now using (9), increment (10) can be written as

$$\Delta S(v)=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left[R(x,y,z)+(\omega_{1,1,1}f)(x,y,z)\right]\Delta u_{xyz}(x,y,z)dxdydz-$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}f_{1,1,1}(x,y,z)\Delta\varphi(x,y,z)dxdydz. \qquad (11)$$

Obviously, equality (11) is true for all $f\in E_q^{(1,1,1)}$. The expression of operator $\omega_{1,1,1}$ depends on only one element f, i.e. on $f_{1,1,1}$:

$$(\omega_{1,1,1}f)(x,y,z)\equiv f_{1,1,1}(x,y,z)+$$

$$+\int_{x}^{x_1}\int_{y}^{y_1}\int_{z}^{z_1}f_{1,1,1}(\alpha,\beta,\gamma)[A_{0,0,0}(\alpha,\beta,\gamma)+\int_{x_0}^{\alpha}\int_{y_0}^{\beta}\int_{z_0}^{\gamma}K_{0,0,0}(\tau,\xi,\eta;\alpha,\beta,\gamma)d\tau d\xi d\eta]d\alpha d\beta d\gamma+$$

$$+\int_{y}^{y_1}\int_{z}^{z_1}f_{1,1,1}(x,\beta,\gamma)[A_{1,0,0}(x,\beta,\gamma)+\int_{x_0}^{x}\int_{y_0}^{\beta}\int_{z_0}^{\gamma}K_{1,0,0}(\tau,\xi,\eta;x,\beta,\gamma)d\tau d\xi d\eta]d\beta d\gamma+$$

$$+\int_{x}^{x_1}\int_{z}^{z_1}f_{1,1,1}(\alpha,y,\gamma)[A_{0,1,0}(\alpha,y,\gamma)+\int_{x_0}^{\alpha}\int_{y_0}^{y}\int_{z_0}^{\gamma}K_{0,1,0}(\tau,\xi,\eta;\alpha,y,\gamma)d\tau d\xi d\eta]d\alpha d\gamma+$$

$$+\int_{x}^{x_1}\int_{y}^{y_1}f_{1,1,1}(\alpha,\beta,z)[A_{0,0,1}(\alpha,\beta,z)+\int_{x_0}^{\alpha}\int_{y_0}^{\beta}\int_{z_0}^{z}K_{0,0,1}(\tau,\xi,\eta;\alpha,\beta,z)d\tau d\xi d\eta]d\alpha d\beta+$$

$$+\int_{z}^{z_1}f_{1,1,1}(x,y,\gamma)[A_{1,1,0}(x,y,\gamma)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{\gamma}K_{1,1,0}(\tau,\xi,\eta;x,y,\gamma)d\tau d\xi d\eta]d\gamma+$$

$$+\int_{x}^{x_1}f_{1,1,1}(\alpha,y,z)[A_{0,1,1}(\alpha,y,z)+\int_{x_0}^{\alpha}\int_{y_0}^{y}\int_{z_0}^{z}K_{0,1,1}(\tau,\xi,\eta;\alpha,y,z)d\tau d\xi d\eta]d\alpha+$$

$$+\int_{y}^{y_1}f_{1,1,1}(x,\beta,z)[A_{1,0,1}(x,\beta,z)+\int_{x_0}^{x}\int_{y_0}^{\beta}\int_{z_0}^{z}K_{1,0,1}(\tau,\xi,\eta;x,\beta,z)d\tau d\xi d\eta]d\beta.$$

Therefore, equality (11) is true for all $f_{1,1,1}\in L_q(G)$. To simplify expression (11), we introduce the equation

$$(\omega_{1,1,1}f_{1,1,1})(x,y,z)+R(x,y,z)=0,\quad (x,y,z)\in G, \qquad (12)$$

which we call the adjoint equation for optimal control problem (1)-(3).

Now we choose the function $f_{1,1,1}(x,y,z)$ as the solution of equation (12) in $L_q(G)$. Then formula (11) will take a simple form:

$$\Delta S(v) = -\int_{\sqrt{x_0 x_1}}^{x_1}\int_{\sqrt{y_0 y_1}}^{y_1}\int_{\sqrt{z_0 z_1}}^{z_1} f_{1,1,1}(x,y,z)\Delta\varphi(x,y,z)dxdydz. \quad (13)$$

Now consider the following needle variation of the admissible control $v(x,y,z)$:

$$\Delta v_\varepsilon(x,y,z) = \begin{cases} \hat{v} - v(x,y,z), & (\tau,\xi,\eta)\in G_\varepsilon \\ 0, & (x,y,z)\in G\setminus G_\varepsilon \end{cases} \quad (14)$$

where $\hat{v} \in U_\partial$ and $(\tau,\xi,\eta)\in G$ are fixed points, $\varepsilon > 0$ is a sufficiently small parameter, and

$$G_\varepsilon = \left(\tau - \frac{\varepsilon}{2}, \tau + \frac{\varepsilon}{2}\right)\times\left(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}\right)\times\left(\eta - \frac{\varepsilon}{2}, \eta + \frac{\varepsilon}{2}\right).$$

The control $v_\varepsilon(x,y,z)$ defined by the equality

$$v_\varepsilon(x,y,z) = v(x,y,z) + \Delta v_\varepsilon(x,y,z)$$

is an admissible control for all sufficiently small $\varepsilon > 0$ and all $\hat{v} \in U_\partial$, $(\tau,\xi,\eta)\in G$.

This control is called the needle perturbation of the given control $v(x,y,z)$. Obviously,

$$S(v_\varepsilon) - S(v) =$$
$$= -\iiint_{G_\varepsilon} f_{1,1,1}(x,y,z)[\varphi(x,y,z,v(x,y,z)+\Delta v_\varepsilon(x,y,z)) - \varphi(x,y,z,v(x,y,z))]dxdydz =$$
$$= -\iiint_{G_\varepsilon} f_{1,1,1}(x,y,z)[\varphi(x,y,z,\hat{v}) - \varphi(x,y,z,v(x,y,z))]dxdydz. \quad (15)$$

Since the optimal control problem is linear, it follows from (15) the following

Theorem. Suppose $f_{1,1,1}(x,y,z)\in L_q(G)$ is the solution of adjoint equation (12). Then, for the optimality of some admissible control $v(x,y,z)$, it is necessary and sufficient that the maximum condition

$$\max_{\hat{v}\in U_\partial} H(x,y,z,f_{1,1,1}(x,y,z),\hat{v}) = H(x,y,z,f_{1,1,1}(x,y,z),v(x,y,z))$$

held almost for all $(x,y,z)\in G$, where $H(x,y,z,f_{1,1,1},v) = f_{1,1,1}.\varphi(x,y,z,v)$ is a Hamilton-Pontryagin function.

Proof. If the control $v(x,y,z)$ from U_∂ affords the least value to functional (3), then from (15), we have

$$-\iiint_{G_\varepsilon}[H(x,y,z,f_{1,1,1,1}(x,y,z),\hat{v}) - H(x,y,z,f_{1,1,1,1}(x,y,z),v(x,y,z)]dxdydz \ge 0, \quad (16)$$

Dividing both sides of equation (16) by ε^3 and proceeding to the limit at$\varepsilon \to +0$, we get

$$H(\tau,\xi,\eta,f_{1,1,1}(\tau,\xi,\eta),v(\tau,\xi,\eta)) - H(\tau,\xi,\eta,f_{1,1,1}(\tau,\xi,\eta),\hat{v}) \ge 0, \quad (17)$$

almost for all $(\tau,\xi,\eta)\in G$ and for all $v\in U_\partial$. Then, for the optimality of the control $v(x,y,z)\in U_\partial$ it is necessary that condition (17) held.

Besides, equality

$$\Delta S(v) = -\iiint_{G_\varepsilon}\Delta H(x,y,z,f_{1,1,1}(x,y,z),v(x,y,z))dxdydz$$

shows that the fulfillment of this condition is also sufficient for the optimality of the control $v(x,y,z)$, where $\Delta H(x,y,z,f_{1,1,1},v) = H(x,y,z,f_{1,1,1},v+\Delta v) - H(x,y,z,f_{1,1,1},v)$.

The theorem is proved.

This theorem shows that to solve the optimal control problem (1)-(3), it is enough to find the solution $f_{1,1,1}(x,y,z)\in L_q(G)$ of integral equation (12). Then the optimal control $v(x,y,z)$ can be found as a point from U_∂, which affords the maximum value to the function

$H\big(x, y, z\,, f_{1,1,1}(x, y, z), \nu\big)$ on U_∂ relative to ν.

It should be noted that adjoint equation (12) introduced in the study for optimal control problem (1)-(3) is more natural than adjoint problems of the classical form.

Equation (12) is a three-dimensional integral equation. In the particular case when $K_{i,j,k}(\tau, \xi, \eta; x, y, z) \equiv 0$, and the functions $A_{i,j,k}(x, y, z)$ are sufficiently smooth (t i.e. have summable derivatives $D_x^i D_y^j D_z^k A_{i,j,k}(x, y, z)$, then from equation (12) we can also proceed to the adjoint problem of the classical form, defined by formally adjoint differential operator.

4. Conclusion

1. The obtained results can be used to study local and nonlocal optimal control problems associated with nonlocal problems [23, p. 74; 24, p.133] for 3D Bianchi equations with nonsmooth coefficients.

2. The obtained results can also be used in a similar way to obtain the necessary optimality conditions in nonlinear problems of optimal control of systems described by nonlinear 3D Bianchi equations of similar types.

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