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On one linear variable structure optimal control problem

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ABSTRACT

One variable structure optimal control problem described by a set of systems of differential and integral equations and a linear quality criterion is considered. The necessary and sufficient optimality condition is proved.

1. Introduction

Currently, the theory of necessary optimality conditions for various classes of problems of optimal control of ordinary dynamical systems is sufficiently developed (e.g. [1–4]).

The study of a number of control processes, the choice of program controls comes down to the management of step systems described in different time intervals by various equations [5-9]. Such optimal control problems are also called optimal control problems with a variable structure (e.g. [7, 8]).

In [5–9] and others, a number of variable structure optimal control problems described by differential equations were studied and various necessary optimality conditions were established in various ways under various assumptions. In this paper, one variable structure optimal control problem described by a set of systems of differential and integral equations with a linear multipoint quality functional is investigated. It is proved that in the case under consideration an analog of the Pontryagin maximum principle is not only a necessary, but also a sufficient optimality condition.

2. Problem statement

Let us consider a controlled process described by a set of differential and integral equations of the form

$$\dot{x} = A(t)\,x + f(t,u)\,, \quad t \in T_1 = [t_0, t_1]\,, \tag{1}$$

$$x(t_0) = x_0\,, \tag{2}$$

$$y(t) = \int_{t_0}^{t} \Big(B(t,\tau)\,y(\tau) + g\big(t,\tau,u(\tau)\big)\Big)\,d\tau + C\,x(t_1)\,, \quad t \in T_2 = [t_1, t_2]\,. \tag{3}$$

Here, $t_0,\ t_1,\ t_2$ are prescribed $(t_0 < t_1 < t_2)$, x_0 is a prescribed n-dimensional constant vector, $A(t)$ is a prescribed $(n \times n)$ continuous matrix function, $f(t,u)$ is a prescribed n-dimensional

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vector function, $B(t,\tau)$ is a prescribed $(m \times m)$ continuous matrix function, C is a prescribed continuous matrix of corresponding dimensionality, $g(t,\tau,u)$ is a prescribed m-dimensional vector function that is continuous in the set of variables, $u(t)$ $\big(v(t)\big)$ is r (q)-dimensional piecewise continuous vector function with a finite number of points of discontinuity of the first kind with values from prescribed nonempty and bounded sets U (V), i.e.

$$u(t) \in U \subset R^r, \quad t \in T_1, \qquad (4)$$
$$v(t) \in V \subset R^q, \quad t \in T_2.$$

The pair $\big(u(t),v(t)\big)$ with the above properties shall be called an admissible control. It is assumed that for each given admissible control$\big(u^o(t),v^o(t)\big)$ Cauchy problem (1)-(2) has the unique piecewise-smooth solution $x^o(t)$, and Volterra-type integral equation (3) has the unique continuous solution $y^o(t)$.

Suppose $\eta_i,\ i=\overline{1,k}$ $(t_0 < \eta_1 < \eta_2 < \cdots < \eta_k \le t_1)$ and $\xi_i,\ i=\overline{1,k}$ $(t_1 < \xi_1 < \xi_2 < \cdots < \xi_k \le t_2)$ are prescribed.

Let us consider the problem of the minimum of the multipoint functional

$$J(u,v) = \sum_{i=1}^{k} c_i'\, x(\eta_i) + \sum_{i=1}^{k} d_i'\, y(\xi_i)\,, \qquad (5)$$

with constraints (1)-(4).

Here, c_i (d_i), $i=\overline{1,k}$ are the prescribed constant n (m)-dimensional vectors.

The admissible control $\big(u^o(t),v^o(t)\big)$ that satisfies the minimum of functional (5) with constraints (1)-(4) shall be called an optimal control, and the corresponding process $\big(u^o(t),v^o(t),x^o(t),y^o(t)\big)$ – an optimal process. It is assumed that an optimal control exists in the problem under investigation.

3. The necessary and sufficient optimality condition

Suppose $\big(u^o(t),v^o(t),x^o(t),y^o(t)\big)$ is a fixed admissible process and the set. Let us denote by $\big(\bar{u}(t)=u^o(t)+\Delta u(t),\ \bar{v}(t)=v^o(t)+\Delta v(t),\ \bar{x}(t)=x^o(t)+\Delta x(t),\ \bar{y}(t)=y^o(t)+\Delta y(t)\big)$ arbitrary admissible process and write the increment of the quality functional

$$J(\bar{u},\bar{v}) - J(u^o,v^o) = \sum_{i=1}^{k} c_i'\,\Delta x(\eta_i) + \sum_{i=1}^{k} d_i'\,\Delta y(\xi_i)\,. \qquad (6)$$

It is clear that the increment $\big(\Delta x(t),\Delta y(t)\big)$ of the trajectory will be the solution to the system of equations

$$\Delta\dot{x}(t) = A(t)\,\Delta x(t) + \Big(f\big(t,\bar{u}(t)\big) - f\big(t,u(t)\big)\Big)\,, \qquad (7)$$
$$\Delta x(t_0) = 0\,,$$

$$\Delta y(t) = \int_{t_1}^{t} \Big(B(t,\tau)\,\Delta y(\tau) + \Big(g\big(t,\tau,\bar{v}(\tau)\big) - g\big(t,\tau,v^o(\tau)\big)\Big)\Big)\,d\tau + C\,\Delta x(t_1)\,. \qquad (8)$$

Suppose $\psi^o(t)$, $p^o(t)$ are still arbitrary n and m-dimensional vector functions. From identities (7) and (8), we get

$$\int_{t_0}^{t_1} \psi^{o'}(t)\,\Delta\dot{x}(t)\,dt = \int_{t_0}^{t_1} \psi^{o'}(t)\,A(t)\,\Delta x(t)dt + \int_{t_0}^{t_1} \psi^{o'}(t)\,\Big(f\big(t,\bar{u}(t)\big) - f\big(t,u^o(t)\big)\Big)\,dt\,, \qquad (9)$$

$$\int_{t_1}^{t_2} p^{o\prime}(t)\,\Delta y(t)\,dt = \int_{t_1}^{t_2}\left[\int_{t_1}^{t} p^{o\prime}(t)\left[B(t,\tau)\,\Delta y(\tau) + \left(g\big(t,\tau,\bar{v}(\tau)\big) - g\big(t,\tau,v^o(\tau)\big)\right)\right]d\tau\right]dt + $$
$$+\int_{t_1}^{t_2} p^{o\prime}(t)\,\mathrm{C}\,\Delta x(t_1)\,dt. \tag{10}$$

Given identities (9) and (10), in the formula of increment (6) we get

$$\Delta J(u^o,v^o) = \sum_{i=1}^{k} c_i'\,\Delta x(\eta_i) + \sum_{i=1}^{k} d_i'\,\Delta y(\xi_i) + \int_{t_0}^{t_1}\psi^{o\prime}(t)\,\Delta\dot{x}(t)\,dt - \int_{t_0}^{t_1}\big(A'(t)\,\psi(t)\big)'\,\Delta x(t)dt -$$
$$-\int_{t_0}^{t_1}\psi^{o\prime}(t)\,\left(f\big(t,\bar{u}(t)\big) - f\big(t,u^o(t)\big)\right)dt + \int_{t_1}^{t_2} p^{o\prime}(t)\,\Delta y(t)\,dt - \tag{11}$$
$$-\int_{t_1}^{t_2}\left[\int_{t}^{t_2} p^{o\prime}(\tau)\left[B(\tau,t)\,\Delta y(t) + \left(g\big(\tau,t,\bar{v}(t)\big) - g\big(\tau,t,v^o(t)\big)\right)\right]d\tau\right]dt +$$
$$+\int_{t_1}^{t_2} p^{o\prime}(t)\,C\,\Delta x(t_1)\,dt.$$

Suppose $\alpha_i(t)$ is the characteristic function of the segment $[t_0,\eta_i]$, and $\beta_i(t)$ is the characteristic function of the segment $[t_1,\xi_i]$.

It is clear that

$$\Delta x(\eta_i) = \int_{t_0}^{t_1}\alpha_i(t)\,\Delta\dot{x}(t)\,dt. \tag{12}$$

Further, it follows from (8) that

$$\Delta y(\xi_i) = \int_{t_1}^{\xi_i}\left(B(\xi_i,\tau)\,\Delta y(\tau) + \left(g\big(\xi_i,\tau,\bar{v}(\tau)\big) - g\big(\xi_i,\tau,v^o(\tau)\big)\right)\right)d\tau + \mathrm{C}\,\Delta x(t_1) = \tag{13}$$
$$= \int_{t_1}^{t_2}\beta_i(\tau)\left(B(\xi_i,\tau)\,\Delta y(\tau) + \left(g\big(\xi_i,\tau,\bar{v}(\tau)\big) - g\big(\xi_i,\tau,v^o(\tau)\big)\right)\right)d\tau + \mathrm{C}\int_{t_0}^{t_1}\Delta\dot{x}(t)\,dt\,.$$

Taking into account identities (12), (13), in (11) we have

$$\Delta J(u^o,v^o) = \int_{t_0}^{t_1}\sum_{i=1}^{k}\alpha_i(t)\,c_i'\,\Delta\dot{x}(t)\,dt + \int_{t_0}^{t_1}\psi^{o\prime}(t)\,\Delta\dot{x}(t)dt$$
$$-\int_{t_0}^{t_1}\left(\int_{t}^{t_1}\big(A'(\tau)\,\psi^0(\tau)\big)'d\tau\right)\Delta\dot{x}(t)dt -$$
$$-\int_{t_0}^{t_1}\psi^{o\prime}(t)\,\left(f\big(t,\bar{u}(t)\big) - f\big(t,u^o(t)\big)\right)dt + \sum_{i=1}^{k}\int_{t_1}^{t_2}\beta_i(t)\,d_i'\,B(\xi_i,t)\,\Delta y(t)dt + \tag{14}$$

$$+\sum_{i=1}^{k}\int_{t_1}^{t_2}\beta_i(\tau)\,d_i'\,\left(g\left(\xi_i,\tau,\bar{v}(\tau)\right)-g\left(\xi_i,\tau,v^o(\tau)\right)\right)d\tau+\sum_{i=1}^{k}d_i'\,C\int_{t_0}^{t_1}\alpha_i(t)\,\Delta\dot{x}(t)\,dt+$$

$$+\int_{t_1}^{t_2}p^{o'}(t)\,\Delta y(t)\,dt-\int_{t_1}^{t_2}\left[\int_{t}^{t_2}p^{o'}(\tau)B(\tau,t)d\tau\right]\Delta y(t)\,dt+$$

$$-\int_{t_1}^{t_2}\left[\int_{t}^{t_2}p^{o'}(\tau)\left(g\left(\tau,t,\bar{v}(t)\right)-g\left(\tau,t,v^o(t)\right)\right)d\tau\right]dt-\int_{t_1}^{t_2}\left[p^{o'}(t)\,\mathrm{C}\int_{t_0}^{t_1}\Delta\dot{x}(\tau)\,d\tau\right]dt.$$

Suppose $\psi^o(t)$ and $p^o(t)$ satisfy the relations

$$\psi^o(t)=-\sum_{i=1}^{k}\alpha_i(t)\,c_i+\int_{t}^{t_1}A'(\tau)\,\psi(\tau)\,d\tau+\int_{t_1}^{t_2}\mathrm{C}'p^o(\tau)\,d\tau-\sum_{i=1}^{k}\mathrm{C}'\,d_i\alpha_i(t)\,, \tag{15}$$

$$p^o(t)=\int_{t}^{t_2}B'(\tau,t)\,p^o(\tau)\,d\tau+\sum_{i=1}^{k}\beta_i(t)\,d_iB(\xi_i,t)\;. \tag{16}$$

Then the formula for increment (14) of the quality functional will take the form:

$$\Delta J(u^o,v^o)=-\int_{t_0}^{t_1}\psi^{o'}(t)\left(f\left(t,\bar{u}(t)\right)-f\left(t,u^o(t)\right)\right)dt+$$

$$+\sum_{i=1}^{k}\int_{t_1}^{t_2}\beta_i(t)\,d_i'\,\left(g\left(\xi_i,t,\bar{v}(t)\right)-g\left(\xi_i,t,v^o(t)\right)\right)dt-$$

$$-\int_{t_1}^{t_2}\left[\int_{t}^{t_2}p^{o'}(\tau)\left(g\left(\tau,t,\bar{v}(t)\right)-g\left(\tau,t,v^o(t)\right)\right)d\tau\right]dt\,.$$

If we introduce the notation

$$H\left(t,u(t),\psi^o(t)\right)=\psi_o'(t)\,f\left(t,u(t)\right),$$

$$M\left(t,v(t),p^o(t)\right)=\int_{t}^{t_2}p^{o'}(\tau)g\left(\tau,t,v(t)\right)d\tau-$$

$$-\sum_{i=1}^{k}\beta_i(t)\,d_i'\,g\left(\xi_i,t,v(t)\right),$$

hence we get

$$\Delta J(u^o,v^o)=-\int_{t_0}^{t_1}\left(H\left(t,\bar{u}(t),\psi^o(t)\right)-H\left(t,u^o(t),\psi^o(t)\right)\right)dt-$$

$$-\int_{t_1}^{t_2}\left(M\left(t,\bar{v}(t),p^o(t)\right)-M\left(t,v^o(t),p^o(t)\right)\right)dt\,. \tag{17}$$

Note that equations (15), (16) are Volterra-type linear integral equations. We shall call them the adjoint system in the problem under investigation.

The formula of increment (17) allows proving the necessary and sufficient optimality condition in the form of an analogue of the Pontryagin maximum principle. Using McShane needle-like variations [10, 11], we prove

Theorem 1. It is necessary for the optimality of the admissible control $(u^o(t), v^o(t))$ that the relations

$$\max_{u \in U} H(\theta, u, \psi^o(\theta)) = H(\theta, u^o(\theta), \psi^o(\theta)),$$
$$\max_{v \in V} M(\xi, v, p^o(\xi)) = M(\xi, v^o(\xi), p^o(\xi))$$

hold for all $\theta \in [t_0, t_1)$ and $\xi \in [t_1, t_2)$, respectively.

Here, $\theta \in [t_0, t_1)$ $(\xi \in [t_1, t_2))$ are arbitrary points of continuity of the controls $u^o(t)$ and $v^o(t)$, respectively.

A similar result can also be obtained in the case of a nonlinear convex and smooth quality functional using a modified version of the scheme from [11].

4. Conclusion

In this paper, we consider one variable structure optimal control problem described by differential equations and a Volterra-type integral equation in various time intervals. Under the assumption that the equations under investigation are linear in the state vector and quality criterion, using the specifics of the problem under investigation, an adjoint system is introduced in the form of Volterra-type integral equations. A formula for the increment of the quality criterion is constructed, and with it the necessary and sufficient optimality condition in the form of the Pontryagin maximum principle is proved.

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