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Mangeron's equation in the semi-Markov random walk process

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A B S T R A C T

This paper considers the sequence $\{\xi_k^+, \eta_k^+, \xi_k^-\}_{k=1}^{\infty}$ of independent identically distributed, positive, independent random variables and the sequence $\{\eta_k^-\}_{k=1}^{\infty}$ of negative random variables. On the basis of these random variables, a semi-Markov random walk process with a delaying screen at zero is constructed, and an integral equation for the conditional distribution $R(t,x|z,h)$ of this process is found using the formula of total probability. In the class of distributions decreasing exponentially fast, using the method of successive Laplace integral transforms in time t and Laplace-Stiltes in phase x, this integral equation is reduced to a partial differential equation – to the fourth-order Mangeron equation. Thus, it is proved that the double integral image $\tilde{\tilde{R}}(\theta,\alpha|z,h)$ of the conditional distribution of the constructed semi-Markov random walk process is a solution to the obtained Mangeron equation.

1. Introduction

To study the distribution of a semi-Markov random walk and its main boundary functionals, some authors used asymptotic, factorization, and other methods [1-5]. In this paper, narrowing the class of the random walk, the integral equation for the Laplace transform in time, the Laplace-Stieltjes transform in the phase of the conditional distribution of the semi-Markov random walk process are reduced to the Mangeron equation [6].

2. Probabilistic problem statement

Suppose the sequence $\{\xi_\kappa^+, \eta_k^+, \xi_\kappa^-\}_{k=\overline{1,\infty}}$ is given on the probability space $(\Omega, F, P(\cdot))$ of independent identically distributed, positive, independent random variables and the negative random variable η$\eta_k^- < 0. k = \overline{1,\infty}$.

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Using these random variables, we construct the following random processes

$$X^{\pm}(t)=\sum_{i=1}^{k-1}\eta_i^{\pm},\ \text{ if }\ \sum_{i=1}^{k-1}\xi_i^{\pm}\le t<\sum_{i=1}^{k}\xi_i^{\pm},\ k=\overline{1,\infty}.$$

We can write these processes in the following form:

$$X^{\pm}(t)=\sum_{i=1}^{\vartheta^{\pm}(t)}\eta_i^{\pm},\ \text{where } v^{\pm}(t)=k, \text{if} \sum_{i=1}^{k-1}\xi_i^{\pm}\le t<\sum_{i=1}^{k}\xi_i^{\pm}.$$

Let us call the process $X_1(t)=X^+(t)-X^-(t)$ a complex semi-Markov random walk process.

Denote

$$\tau_k^{\pm}=\sum_{i=1}^{k}\xi_i^{\pm},\quad k=\overline{1,\infty}$$

We arrange these random variables in ascending order

$$\{\tau_k\},\ \ k=\overline{1,\infty}.$$

Denote

$$\eta_k=\begin{cases}\eta_i^+, & \text{if } \tau_k=\tau_i^+,\\ \eta_j^-, & \text{if } \tau_k=\tau_j^-.\end{cases}$$

Construct the following process [1, c.51]

$$X(t)=\varsigma_{k,}\ \text{ if }\ \tau_k\le t<\tau_{k+1},\quad k=\overline{1,\infty},$$

$$\varsigma_0=z,\quad \varsigma_k=max(0,\varsigma_{k-1}+\eta_k),\ z>0$$

Let us call the process $X(t)$ differential with a random walk and with a delaying screen at zero. The aim of this paper is to study the distribution of this semi-Markov process.

3. Solution

Note that neither the moments τ_k^+,, nor the moments τ_k^- are Markov moments. If we know the value of the process $X(t)$ at the moment *t*, then to determine the further behavior of the process, we also need to know when a positive jump will occur for the first time after τ_k^-. Therefore, when studying the process $X(t)$, it is natural to also consider two following processes.

$$\delta^{\pm}(t)=min\left[\tau_k^{\pm}-t\right]$$

We must investigate the distribution of the process $X(t)$ in the following form

$$P\{X(t)<x|X(0)=z,\delta^+(0)=h\}=P\{X(t)<x|X(0)=z,\xi_1^+=h\}$$

or

$$P\{X(t)<x|X(0)=z,\delta^-(0)=h\}=P\{X(t)<x|X(0)=z,\xi_1^-=h\}.$$

From the total probability formula we have

$$P\{X(t)<x|X(0)=z,\xi_1^+=h\}=P\{X(t)<x,\xi_1^->t\,|X(0)=z,\xi_1^+=h\}+$$
$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{s=0}^{t}P\{\xi_1^-\in\ ds;X(s)\in dy;\ \delta^+(s)\in du|X(0)=z,\xi_1^+=h\}P\{X(t-s)<x|X(0)=$$
$$=y,\xi_1^+=h\}\qquad(1)$$

Denote

$$R(t,x|z,h) = P\{X(t) < x|X(0) = z; \xi_1^+ = h\}.$$

Then (1) takes the form

$$R(t,x|z,h) = P\{X(t) < x, \xi_1^- > t\,|X(0) = z, \xi_1^+ = h\} +$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{s=0}^{t} P\{\xi_1^- \epsilon ds; X(s) \in dy;\ \delta^+(s) \in du|X(0) = z, \xi_1^+ = h\}R(t-s, x|X(0) =$$

$$= y, \xi_1^+ = h)$$

Since $v^+(t) \geq 0$, the last equation can be written in the following form

$$R(t,x|z,h) = P\{z < x, \xi_1^- > t\,, v^+(t) = 0|\xi_1^+ = h\} +$$
$$+P\{z + X^+(t) < x, \xi_1^- > t, v^+(t) > 0|\xi_1^+ = h\} +$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{s=0}^{t} P\{\,\xi_1^- \in ds; s < \xi_1^+; \max(0, z-\eta_1^-)\,\epsilon\, dy; \xi_1^+ - s \in du|\xi_1^+ = h\}R(t-s, x|y,u) +$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{s=0}^{t} P\ \Big\{\xi_1^- \in ds; s > \xi_1^+; \max\Big(0, z + \sum_{i=1}^{v^+(s)} \eta_i^+ - \eta_1^-\Big)\,\epsilon\, dy;$$

$$\sum_{i=1}^{v^+(s)+1} \xi_i^+ - s \in du|\xi_1^+ = h\Big\} R(t-s, x|y,u).$$

Further,

$$\tilde{R}(t,x|z,h) = P\{z < x, \xi_1^- > t\,, \xi_1^+ < t, |\xi_1^+ = h\} +$$

$$+P\{z + X^+(t) < x, \xi_1^- > t, v_1^+(t) > 0|\xi_1^+ = h\} +$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{s=0}^{t} P\ \{\,\xi_1^- \in ds; s < \xi_1^+; \max(0, z-\eta_1^-)\,\epsilon dy; \xi_1^+ - s \in du|\xi_1^+ = h\}R(t-s, x|y,u) +$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{s=0}^{t} P\ \{\xi_1^- \in ds; s > \xi_1^+; \max(0, z + X^+(s) - \eta_1^-)\,\epsilon\, dy;$$

$$\sum_{i=1}^{v^+(s)+1} \xi_i^+ - s \in du|\xi_i^+ = h\Big\} R(t-s, x|y,u).$$

Denote $\tilde{R}(\theta,x|z,h) = \int_{t=0}^{\infty} e^{-\theta t} R(t,x|z,h)dt, \theta > 0$, where $\tilde{R}(\theta,x|z,h)$ is a Laplace transform with respect to time t.

Then the last equation can be written as follows

$$\tilde{R}(\theta,x|z,h) = \varepsilon(x-z)\int_{t=0}^{\infty} e^{-\theta t}\, P\{\,\xi_1^- > t, \xi_1^+ > t|\,\xi_1^+ = h\}dt +$$

$$+\int_{t=0}^{\infty} e^{-\theta t} P\Big\{z + \sum_{i=1}^{v^+(t)} \eta_i^+ < x; \xi_1^- > t\,, v^+(t) > 0, \xi_1^+ < t|\,\xi_1^+ = h\Big\}\,dt +$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty} \tilde{R}\,(\theta,x|y,u)\int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- \in dt; t < \xi_1^+; \max(0, z-\eta_1^-\}\epsilon\, dy; \xi_1^+ - t \in du|\xi_1^+ = h\Big\}$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{R}\,(\theta,x|y,u)\int_{t=0}^{\infty}e^{-\theta t}P\left\{\xi_1^- \in dt; t>\xi_1^+; \max(0,z+\sum_{i=1}^{v^+(t)}\eta_i^+-\eta_i^-)\in dy;\right.$$

$\left.\sum_{i=1}^{v^+(t)+1}\xi_i^+ - t\in du|\xi_1^+=h\right\}$, where $\varepsilon(x-z)=\begin{cases}0, x-z<0\\ 1, x-z>0\end{cases}$

Taking into account that in (1) $\xi_1^+=h$ and the random variables ξ_1^+, ξ_1^- are independent, then the last equation can be written in the following form

$$\tilde{R}(\theta,x|z,h)=\varepsilon(x-z)\int_{t=0}^{h}e^{-\theta t}P\{\xi_i^->t\}dt+$$

$$+\int_{t=0}^{\infty}e^{-\theta t}P\left\{z+\sum_{i=1}^{v^+(t)}\eta_i^+<x; v_1^+(t)>0|\,\xi_1^+=h\right\}P\{\xi_1^->t\}dt+$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{R}\,(\theta,x|y,u)\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^-\in dt; t<\xi_1^+;\xi_1^+-t\,\epsilon\, du|\xi_1^+=h\}P\{max(0,z-\eta_1^-)\in dy\}+$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{R}\,(\theta,x|y,u)\int_{t=0}^{\infty}e^{-\theta t}P\left\{\xi_1^-\in dt; t>\xi_1^+;\ max(0,z+\sum_{i=1}^{v^+(t)}\eta_i^+-\eta_1^-)\in dy;\right.$$

$$\left.\sum_{i=1}^{v^+(t)+1}\xi_i^+-t\in du|\xi_1^+=h\right\}$$

Further,

$$\tilde{R}(\theta,x|z,h)=\varepsilon(x-z)\int_{t=0}^{h}e^{-\theta t}P\{\xi_1^->t\}dt+$$

$$+\int_{t=h}^{\infty}e^{-\theta t}P\left\{\sum_{i=1}^{v^+(t)+1}\eta_i^+<x-z; v^+(t)>0,\xi_1^+=h\right\}P\{\xi_1^->t\}dt+$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{R}\,(\theta,x|y,u)\int_{t=0}^{h}e^{-\theta t}P\{\xi_1^-\in dt\}P\{h-t\in du\}P\{max(0,z-\eta_1^-)\in dy\}+$$

$$+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{R}(\theta,x|y,u)\int_{t=h}^{\infty}e^{-\theta t}d_u d_y P\left\{max\left(0,z+\sum_{i=1}^{v^+(t)}\eta_i^+-\eta_1^-\right)<y\right.$$

$$\left.\sum_{i=1}^{v^+(t)+1}\xi_i^+-t<u|\xi_1^+=h\right\}P\{\xi_1^-\in dt\}$$

After some transformations we get

$$\tilde{R}(\theta,x|z,h)=\varepsilon(x-z)\int_{t=0}^{h}e^{-\theta t}P\{\xi_i^->t\}dt+$$

$$+e^{-\theta h}\int_{v=0}^{\infty} e^{-\theta v}\, P\,\{\eta_i^+ < x-z\} P\{0<v<\xi_1^+\} P\{\xi_1^- > v+h\} +$$

$$+e^{-\theta h}\int_{v=0}^{\infty} e^{-\theta v}\sum_{k=1}^{\infty} P\left\{\sum_{i=1}^{k}\eta_1^+ < x-z\right\} P\left\{\sum_{i=1}^{k-1}\xi_i^+ < v < \sum_{i=1}^{k}\xi_i^+\right\} P\{\xi_1^- > v+h\}dv -$$

$$-e^{-\theta h}P\{\eta_1^- > z\}\quad \int_{v=0}^{h} e^{\theta h}\tilde{R}(\theta,x|0,v)d_{h-v}P\{\xi_1^- < h-v\} -$$

$$-e^{-\theta h}\int_{y=0}^{\infty}\int_{v=0}^{h} e^{\theta v}\,\tilde{R}(\theta,x|y,v)d_{h-v}P\{\xi_1^- < h-v\}d_yP\{\eta_1^- > z-y\} +$$

$$+e^{-\theta h}\int_{u=0}^{\infty}\tilde{R}(\theta,x|0,u)\int_{v=0}^{\infty} e^{-\theta v}duP\{0\le v<\xi_1^- < u+v\}dvP\{\xi_1^- < h+v\}P\{\eta_1^+ - \eta_1^- < -z\} +$$

$$+e^{-\theta h}\int_{u=0}^{\infty}\tilde{R}\,(\theta,x|0,u)\int_{v=0}^{h} e^{-\theta v}du\sum_{k=2}^{\infty} P\left\{\sum_{i=1}^{k-1}\xi_i^+ \le v < \sum_{i=1}^{k}\xi_i^+ < u+v\right\}\tilde{R}(\theta,x|y,z)dvP\{\xi_1^- \le v+h\}\times$$

$$\times P\left\{\textstyle\sum_{i=1}^{k}\eta_i^+ - \eta_1^- < y-z\right\} +$$

$$+e^{-\theta h}\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{R}(\theta,x|y,u)d_yP\,\{\eta_1^+ - \eta_1^- < y-z\}\times$$

$$\times\int_{v=0}^{\infty} e^{-\theta v}d_uP\{0\le v<\xi_1^+ < u+v\}dvP\{\xi_1^- < v+h\} +$$

$$+e^{-\theta h}\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{R}\,(\theta,x|y,u)\int_{v=0}^{\infty} e^{-\theta v}du\sum_{k=2}^{\infty} P\left\{\sum_{i=1}^{k-1}\xi_i^+ \le v < \sum_{i=1}^{k}\xi_i^+ < u+v\right\}dv\times$$

$$\times P\{\xi_1^- < v+h\}d_yP\left\{\sum_{i=1}^{k}\eta_i^+ - \eta_i^- < y-z\right\}$$

Denote:

$$\tilde{\tilde{R}}(\theta,\alpha|z,h) = \int_{x=0}^{\infty} e^{-\alpha x}d_x\tilde{R}(\theta,x|z,h), \qquad \alpha>0.$$

Then we write the last equation in the following form, where $\varphi_{\eta_1^+(\alpha)} = Ee^{-\alpha\eta_1^+}$ is a Laplace transform of the random variable η_1^+:

$$\tilde{\tilde{R}}(\theta,\alpha|z,h) = e^{-\alpha z}\int_{t=0}^{h} e^{-\theta t}P\{\xi_1^- > t\}dt +$$

$$+e^{-\alpha z}e^{-\theta h}\varphi_{\eta_1^+}(\alpha)\int_{v=0}^{\infty} e^{-\theta v}\,P\{\xi_1^+ > v\}P\{\xi_1^- > v+h\}dv +$$

$$+e^{-\alpha z}e^{-\theta h}\sum_{k=2}^{\infty}\varphi_{\eta_1^+}^{k}(\alpha)\int_{v=0}^{\infty}e^{-\theta v}P\{v^+(t)=k-1\}P\{\xi_1^- > v+h\}dv+$$

$$-e^{-\theta h}P\{\eta_1^- > z\}\int_{v=0}^{h}e^{\theta v}\tilde{\tilde{R}}(\theta,\alpha|0,v)d_{h-v}P\{\xi_1^- < h-v\}+$$

$$+e^{-\theta h}\int_{y=0}^{z}\int_{v=0}^{h}e^{\theta u}\tilde{\tilde{R}}(\theta,\alpha|y,v)d_{h-v}P\{\xi_1^- < h-v\}d_yP\{\eta_1^- > z-y\}+$$

$$+e^{-\theta h}\int_{u=0}^{\infty}\tilde{\tilde{R}}(\theta,\alpha|0,u)\int_{v=0}^{h}e^{-\theta v}d_uP\{\xi_1^- < u+v\}d_vP\{\xi_1^- < h+v\}` \times P\{\eta_1^+ - \eta_1^- < -z\}+$$

$$+e^{-\theta h}\int_{u=0}^{\infty}\tilde{\tilde{R}}(\theta,\alpha|o,u)\int_{v=0}^{h}e^{-\theta v}du\sum_{k=2}^{\infty}\int_{g_1=0}^{v}P\{g_1 \le v < g_1+\xi_1^+ < u+v\}dg_1 \times$$

$$\times P\left\{\sum_{i=1}^{k-1}\xi_i^+ \le g_1\right\}dvP\{\xi_1^- < h+v\}dy\int_{g_2=0}^{\infty}P\{g_2-\eta_1^- < y-z\}dg_2P\left\{\sum_{i=1}^{k}\eta_i^+ < g_2\right\}+$$

$$+e^{-\theta h}\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{\tilde{R}}(\theta,\alpha|y,u)d_yP\{\eta_1^+ - \eta_1^- < y-z\}\int_{v=0}^{\infty}e^{-\theta v}du \times$$

$$\times P\{0 \le v < \xi_1^+ < u+v\}dvP\{\xi_1^- < v+h\}$$

$$e^{-\theta h}\int_{y=0}^{\infty}\int_{u=0}^{\infty}\tilde{\tilde{R}}(\theta,\alpha|y,u)\int_{v=0}^{\infty}e^{-\theta v}du\sum_{k=2}^{\infty}\int_{g_1=0}^{v}P\{g_1 \le v < g_1+\xi_1^+ < u+v\}dg_1 \times$$

$$\times P\left\{\sum_{i=1}^{k-1}\xi_i^+ \le g_1\right\}dvP\{\xi_i^- < v+h\}dy \times$$

$$\times \int_{g_2=0}^{\infty}P\{g_2-\eta_1^- < y-z\}dg_2P\left\{\sum_{i=1}^{k}\eta_1^+ < g_2\right\}$$

Let us narrow the class of distributions. Suppose that the random variables $\xi_i^{\pm}$, $\eta_i^{\pm}$ have an Erlang distribution with the parameters $\lambda_{\pm}$ and $\mu_{\pm}$, respectively. Then, after certain calculations, the previous equation with respect to $\tilde{\tilde{R}}$ is reduced to the following equation.

$$\tilde{\tilde{R}}(\theta,\alpha|z,h)=\left[\frac{2\lambda_-+\theta}{(\lambda_-+\theta)^2}+\frac{2\lambda_-+\lambda_-h(\lambda_-+\theta)}{(\lambda_-+\theta)^2}e^{-(\lambda_-+\theta)h}\right]e^{-\alpha z}+$$

$$+e^{\alpha z}e^{-(\lambda_-+\theta)h}\varphi_{\eta_1^+}(\alpha)\left[\frac{1+\lambda_-h}{(\lambda_-+\lambda_++\theta)}+\frac{(\lambda_-+\lambda_++\lambda_+\lambda_-h)}{(\lambda_-+\lambda_++\theta)^2}+\frac{2\lambda_+\lambda_-}{(\lambda_-+\lambda_++\theta)^3}\right]+$$

$$+e^{\alpha z}e^{-(\lambda_-+\theta)h}\left[\frac{(1+\lambda_-h)\left[\varphi_{\eta_1^+}(\alpha)+\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]}{\lambda_+\left[1-\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]+\lambda_-+\theta}+\frac{(1+\lambda_-h)\left[\varphi_{\eta_1^+}(\alpha)-\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]}{\lambda_+\left[1+\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]+\lambda_-+\theta}-\right.$$

$$-\frac{2\lambda_+(1+\lambda_-h)\sqrt{\varphi_{\eta_1^+}(\alpha)}}{(\lambda_-+\lambda_++\theta)^2}+\frac{\lambda_-\left[\varphi_{\eta_1^+}(\alpha)+\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]}{\left\{\lambda_+\left[1-\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]+\lambda_-+\theta\right\}^2}-\frac{\lambda_-\left[\varphi_{\eta_1^+}(\alpha)-\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]}{\left\{\lambda_+\left[1-\sqrt{\varphi_{\eta_1^+}(\alpha)}\right]+\lambda_-+\theta\right\}^2}-$$

$$\left.\frac{2\lambda_+\lambda_-+\theta}{(\lambda_-+\lambda_++\theta)^3}\right]+$$

$$+\lambda_-^2e^{-\mu_-z}e^{-(\lambda_-+\theta)h}(1+\mu_-z)\int_{v=0}^{h}(h-v)e^{(\lambda_-+\theta)h}\tilde{\tilde{R}}(\theta,\alpha|0,v)dv+$$

$$+(\lambda_-\mu_-)^2e^{-\mu_-z}e^{-(\lambda_-+\theta)h}\int_{y=0}^{z}(z-y)\,e^{\mu_-y}\int_{v=0}^{h}(h-v)e^{(\lambda_-+\theta)v}\,\tilde{\tilde{R}}(\theta,\alpha|y,v)dvdy+$$

$$+(\lambda_+\lambda_-)^2e^{-\mu_-z}e^{-(\lambda_-+\theta)h}\{1+\mu_-z-\mu_-^2\left[\tfrac{1-\mu_+z}{\mu_++\mu_-}\left(z+\tfrac{1}{\mu_++\mu_-}\right)+\tfrac{\mu_+}{\mu_++\mu_-}\left[z^2+\tfrac{2}{\mu_++\mu_-}\times\right.\right.$$

$$\left.\left.\times\left(z+\frac{\mu_+}{\mu_++\mu_-}\right)\right]\right]\int_{u=0}^{\infty}e^{-\lambda_+u}\,\tilde{\tilde{R}}(\theta,\alpha|0,u)\left[\frac{2}{(\lambda_++\lambda_-+\theta)^3}+\frac{u+h}{(\lambda_++\lambda_-+\theta)^2}+\right.$$

$$\left.+\frac{uh}{\lambda_++\lambda_-+\theta}\right]dy+\lambda_+^3\lambda_-^2\mu_+e^{-\mu_-z}e^{-(\lambda_-+\theta)h}\int_{u=0}^{\infty}e^{-\lambda_+u}\,\tilde{\tilde{R}}(\theta,\alpha|0,u)\{(1+\mu_-z)\times$$

$$\times\left[\sum_{k=2}^{\infty}(2k-1)2k\,\frac{\lambda_-^{2k-3}\mu_+^{2k-1}}{(\lambda_++\lambda_-+\theta)^{2k+1}(\mu_++\mu_-)^{2k}}\right.$$

$$+(u+h)\sum_{k=2}^{\infty}(2k-1)\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\,\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}+$$

$$+\sum_{k=2}^{\infty}\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k-1}}\,\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}uh+$$

$$+\sum_{k=2}^{\infty}(2k-2)\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\,\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}\,h-$$

$$-\sum_{k=2}^{\infty} 2k(2k-2)\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k+1}}\ \frac{\mu_+^{2k-1}}{(\mu_+ + \mu_-)^{2k}}\Bigg] +$$

$$+2\mu_- \Bigg[\sum_{k=2}^{\infty}(2k-1)(2k)^2\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k}}\frac{\mu_+^{2k-1}}{(\mu_+ + \mu_-)^{2k}} +$$

$$+(u+h)\sum_{k=2}^{\infty}(2k-1)2k\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k}}\ \frac{\mu_+^{2k-1}}{(\mu_+ + \mu_-)^{2k}} + uh\sum_{k=2}^{\infty}2k\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k-1}}\times$$

$$\times\frac{\mu_+^{2k-1}}{(\mu_+ + \mu_-)^{2k+1}} - h\sum_{k=2}^{\infty}2k(2k-2)\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k}}\ \frac{\mu_+^{2k-1}}{(\mu_+ + \mu_-)^{2k+1}} - \sum_{k=2}^{\infty}(2k-1)(2k)^2\times$$

$$\times\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k+1}}\ \frac{\mu_+^{2k-1}}{(\mu_+ + \mu_-)^{2k+1}}\Bigg]\Bigg\} + (\lambda_+\lambda_-\mu_+\mu_-)^2 e^{\mu_+ z} e^{-(\lambda_- + \theta)h}\int_{y=z}^{\infty} e^{-\mu_+ y}\times$$

$$\times\int_{u=0}^{\infty} e^{-\lambda_+ u}\ \tilde{\tilde{R}}(\theta,\alpha|y,u)\left[\frac{y-z}{(\mu_+ + \mu_-)^2} + \frac{2}{(\mu_+ + \mu_-)^3}\right]dy\left[\frac{2}{(\lambda_+ + \lambda_- + \theta)^3} + \frac{u+h}{(\lambda_+ + \lambda_- + \theta)^2} +\right.$$

$$\left.+\frac{uh}{\lambda_+ + \lambda_- + \theta}\right]du +$$

$$+(\lambda_+\lambda_-\mu_+\mu_-)^2 e^{-\mu_- z} e^{-(\lambda_- + \theta)h}\int_{y=0}^{z} e^{-\mu_- y}\int_{u=0}^{\infty} e^{-\lambda_+ u}\ \tilde{\tilde{R}}(\theta,\alpha|y,u)\left[\frac{y-z}{(\mu_+ + \mu_-)^2}\times\right.$$

$$\times\left(z - y + \frac{1}{\mu_+ + \mu_-}\right) + \frac{1}{\mu_+ + \mu_-}[(z-y)^2 +$$

$$\left.\left.+\frac{2}{\mu_+ + \mu_-}\ \left(z - y + \frac{1}{\mu_+ + \mu_-}\right)\right]\right]\left[\frac{2}{(\lambda_+ + \lambda_- + \theta)^3} + \frac{u+h}{(\lambda_+ + \lambda_- + \theta)^2}\frac{uh}{\lambda_+ + \lambda_- + \theta}\right]dudy +$$

$$+\lambda_+^3\lambda_-^2\mu_+\mu_-^2 e^{-\mu_- z} e^{-(\lambda_- + \theta)h}$$

$$\int_{y=0}^{z} e^{\mu_- y}\int_{u=0}^{\infty} e^{-\lambda_+ u}\ \tilde{\tilde{R}}(\theta,\alpha|y,u)\left\{\sum_{k=2}^{\infty}(2k)^2(2k-1)\frac{\lambda_+^{2k-3}\mu_+^{2k-1}}{(\lambda_+ + \lambda_- + \theta)^{2k+1}(\mu_+ + \mu_-)^{2k+1}}\right\}dudz +$$

$$+(u+h)\sum_{k=2}^{\infty}2k(2k-1)\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k+1}}\ \frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k+1}} -$$

$$-\sum_{k=2}^{\infty}(2k)^2(2k-2)\frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k+1}}\ \frac{\lambda_+^{2k-3}}{(\lambda_+ + \lambda_- + \theta)^{2k+1}}$$

$$-h\sum_{k=2}^{\infty}(2k-2)2k\frac{\lambda_+^{2k-3}\mu_+^{2k-1}}{(\lambda_+ + \lambda_- + \theta)^{2k}(\mu_+ + \mu_-)^{2k+1}} +$$

$$+uh\sum_{k=2}^{\infty}2k\frac{\lambda_+^{2k-3}\mu_+^{2k-1}}{(\lambda_+ + \lambda_- + \theta)^{2k-1}(\mu_+ + \mu_-)^{2k+1}}\Bigg]$$

$$+(z-y)\left[\sum_{k=2}^{\infty}(2k-1)2k\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k+1}}\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}+\right.$$

$$+(u+h)\sum_{k=2}^{\infty}(2k-1)\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}-$$

$$-\sum_{k=2}^{\infty}(2k-1)2k\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k+1}}\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}+h\sum_{k=2}^{\infty}(2k-2)\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}+$$

$$\left.\left.+uh\sum_{k=2}^{\infty}\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\frac{\mu_+^{2k-1}}{(\mu_++\mu_-)^{2k}}\right]\right\}+\lambda_+^3\lambda_-^2\mu_+\mu_-^2e^{-\mu_-z}e^{-(\lambda_-+\theta)h}\times$$

$$\times\int_{y=z}^{\infty}e^{\mu_-y}\int_{u=0}^{\infty}e^{-\lambda_+u}\,\tilde{\tilde{R}}(\theta,\alpha|y,u)\,(z-y)\int_{g_2=y-z}^{\infty}e^{-(\mu_++\mu_-)g_2}\times$$

$$\times\left[\sum_{k=2}^{\infty}2k(2k-1)\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k+1}}\frac{(\mu_+g_2)^{2k-1}}{(2k-1)!}+\right.$$

$$+(u+h)\sum_{k=2}^{\infty}(2k-1)\times\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\frac{(\mu_+g_2)^{2k-1}}{(2k-1)!}+h\sum_{k=2}^{\infty}(2k-2)\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\frac{(\mu_+g_2)^{2k-1}}{(2k-1)!}+$$

$$\left.+uh\sum_{k=2}^{\infty}\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k-1}}\frac{(\mu_+g_2)^{2k-1}}{(2k-1)!}-\sum_{k=2}^{\infty}2k(2k-2)\frac{\lambda_+^{2k-1}}{(\lambda_++\lambda_-+\theta)^{2k+1}}\frac{(\mu_+g_2)^{2k-1}}{(2k-1)!}\right]dg_2+$$

$$+\int_{g_2=y-z}^{\infty}e^{-(\mu_++\mu_-)g_2}\left[\sum_{k=2}^{\infty}2k(2k-1)!\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k+1}}\frac{\mu_+^{2k-1}g_2^{2k}}{(2k-1)!}+\right.$$

$$(u+h)\sum_{k=2}^{\infty}(2k-1)\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\frac{\mu_+^{2k-1}g_2^{2k}}{(2k-1)!}-h\sum_{k=2}^{\infty}\frac{\lambda_+^{2k-3}}{(\lambda_++\lambda_-+\theta)^{2k}}\frac{\mu_+^{2k-1}g_2^{2k}}{(2k-1)!}+$$

$$\left.\left.+uh\sum_{k=2}^{\infty}\frac{\lambda_+^{2k-3}\mu_+^{2k-1}g_2^{2k}}{(\lambda_++\lambda_-+\theta)^{2k}(2k-1)!}-2k(2k-2)\frac{\lambda_+^{2k-3}\mu_+^{2k-1}g_2^{2k}}{(\lambda_++\lambda_-+\theta)^{2k+1}(2k-1)!}\right]dg_2\right\}$$

Further, we perform the following operations:

1) both sides of this integral equation are multiplied by $e^{(\lambda_-+\theta)h}$,
2) the resulting equation is twice differentiated with respect to *h*,,
3) both sides of the resulting equation are divided by $e^{(\lambda_-+\theta)h}$,
4) both sides of the resulting equation are multiplied by e^{μ_-z},
5) the resulting equation is twice differentiated with respect to *z*,
6) both sides of the resulting equation are divided by e^{μ_-z}.

After these operations, we obtain the following partial differential equation:

$$\frac{\partial^4 \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial z^2 \partial h^2} + 2(\lambda_- + \theta)\frac{\partial^3 \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial z^2 \partial h} + 2\mu_- \frac{\partial^3 \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial z \partial h^2} +$$
$$+4(\lambda_- + \theta)\mu_- \frac{\partial^2 \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial z \partial h} + \mu_-^2 \frac{\partial^2 \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial h^2} + (\lambda_- + \theta)\frac{\partial^2 \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial z^2} +$$
$$+2(\lambda_- + \theta)\mu_-^2 \frac{\partial \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial h} + (\lambda_- + \theta)^2\mu_- \frac{\partial \tilde{\tilde{R}}(\theta,\alpha|z,h)}{\partial z} + [2\lambda_-(\lambda_- + \theta) + \theta^2] \times$$
$$\times \mu_-^2 \tilde{\tilde{R}}(\theta,\alpha|z,h) = (\alpha - \mu_-)^2(2\lambda_- + \theta)e^{-\alpha z}.$$

Thus, we have proved that the double integral image $\tilde{\tilde{R}}(\theta,\alpha|z,h)$ of the conditional distribution of the investigated semi-Markov random walk process is a solution of the fourth-order Mangeron equation obtained above.

4. Conclusion

It has been proven in the paper that after the double integral transformation - the Laplace transform in time and the Laplace-Stieltjes transformation in phase, a function has been obtained that depends on the variables z and h and on the transformation parameters θ and α, which satisfies a certain fourth-order Mangeron equation in the variables z and h with respect to $\tilde{\tilde{R}}(\theta,\alpha|z,h)$.

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