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Optimization of capacities and placement of locally distributed sources in feedback control of the heating of the rod

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A B S T R A C T

The problem of synthesis of capacity control and optimal placement of locally distributed heat sources and measuring points during the heating of the rod is investigated. The current values of the controls are determined depending on the temperature values of the rod in the neighborhood of the measurement points. Formulas are obtained for the components of the gradient of the objective functional with respect to the feedback parameters and the coordinates of the location of sources, which can be used in the numerical solution of problems using first-order numerical optimization methods.

1. Introduction

The problem of optimal synthesis of control of locally distributed powers of lumped sources during the heating of the rod is considered. The current values of the capacities of the sources are specified depending on the measured locally averaged values of the temperature of the rod at the points at which the measuring devices are installed. The problem requires optimization of the placement points of sources and the values of the feedback parameters.

This problem belongs to the class of optimal control problems for lumped sources in systems with distributed parameters [1-3], namely, control synthesis problems.

It is necessary to note the increased practical interest specifically in the control problems with feedback with controlled objects, which are called control synthesis problems. In the 1950s, the results of research by L.S. Pontryagin, R.E. Bellman, A.M. Letov and many other scientists made it possible to create real automatic control systems for many different important technological processes and objects [1-9].

The novelty of the results obtained in this study with respect to the considered problem of synthesizing optimal control of the rod heating process by locally distributed lumped sources is related to the following: 1) based on current measurements, the source capacity values are synthesized according to the proposed linear dependence on the measured temperature values at the measurement points; 2) the optimal points of placement of sources are determined; 3) the original problem of synthesis of control of a system with distributed parameters is reduced to a finite-dimensional optimization problem.

In this paper, we obtain formulas for the components of the gradient of the objective functional with respect to the feedback parameters and the coordinates of the location of sources,

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which allow using the well-known standard first-order numerical optimization methods to solve the initial control synthesis problem.

The results of the study can be used to create systems of control and regulation of lumped sources for various other objects with distributed parameters.

2. Problem statement

We study the following problem of rod heating control described by the equation:

$$u_t(x,t) = a^2 u_{xx}(x,t) - \lambda_0[u(x,t) - \theta] + \sum_{i=1}^{N_s} \nu_\varepsilon(x;\eta^i) \int_{t-\Delta t}^{t} q^i(\tau)\mu_{\Delta t}^i(\tau;t)d\tau, \qquad (1)$$

$$(x,t) \in (0,l) \times (0,T],$$

with initial and boundary conditions

$$u(x,t) = b(x) = b = \text{const} \in B,\ \ x \in [0,l],\ \ t \le 0, \qquad (2)$$

$$u_x(0,t) = \lambda[u(0,t) - \theta],\ \ u_x(l,t) = -\lambda[u(l,t) - \theta],\ \ t \in (0,T]. \qquad (3)$$

Here: $u(x,t)$ is the temperature of the rod at the point $x \in [0,l]$ at the time t; a, λ_0, λ are the specified parameters of the heating process; θ is the ambient temperature.

The function $\nu_\varepsilon(x;\tilde{\eta})$, continuously differentiable with respect to $x \in [0,l]$, determines the intensity distribution of the source in the neighborhood $[\tilde{\eta} - \varepsilon, \tilde{\eta} + \varepsilon]$ of its placement point $\tilde{\eta} \in [\varepsilon, l - \varepsilon]$. It has the following properties:

$$\nu_\varepsilon(x;\tilde{\eta}) = \begin{cases} \neq 0 & \text{when } x \in (\tilde{\eta} - \varepsilon, \tilde{\eta} + \varepsilon), \\ = 0 & \text{when } x \notin (\tilde{\eta} - \varepsilon, \tilde{\eta} + \varepsilon), \end{cases}$$

$$\int_{\hat{\eta}-\varepsilon}^{\hat{\eta}+\varepsilon} \nu_\varepsilon(x;\hat{\eta})dx = 1,\ \ \hat{\eta} \in [\varepsilon, l - \varepsilon].$$

Similarly, the function $\mu_{\Delta t}^i(\tilde{\tau};t)$, continuously differentiable with respect to $\tilde{\tau} \in [\Delta t, T]$, determines the intensity distribution of the source $q^i(t)$ at the moment t on the interval $[t - \Delta t, t]$ of the time of exposure $t \in [\Delta t, T]$, and:

$$\mu_{\Delta t}^i(\tilde{\tau};t) = \begin{cases} \neq 0 & \text{when } \tilde{\tau} \in [t - \Delta t, t], \\ = 0 & \text{when } \tilde{\tau} \notin [t - \Delta t, t], \end{cases}$$

$$\int_{t-\Delta t}^{t} \mu_{\Delta t}^i(\tilde{\tau};t)\, d\tilde{\tau} = 1,\ \ i = 1, \dots, N_s.$$

The rod is heated by N_s point sources with controlled powers determined by piecewise-continuous functions $q = q(t) = \left(q^1(t), \dots, q^{N_s}(t)\right)$ such that

$$q^i(t) \in Q^i = \left[\underline{q^i}, \overline{q^i}\right],\ \ i = 1, \dots, N_s,\ \ t \in [0,T), \qquad (4)$$

where $\underline{q^i}$, $\overline{q^i}$, $i = 1, \dots, N_s$ are given. The points $\eta^i \in [\varepsilon, l - \varepsilon]$ determine the coordinates of the location of the i-th source on the rod at the time t, $i = 1, \dots, N_s$.

It is assumed that the initial temperature in (2) at all points of the rod is the same, but it is not specified exactly, and belongs to a given set It is assumed that the initial temperature in (2) at all points of the rod is the same, but it is not specified exactly, and belongs to a given set B with a known density function $\rho_B(b)$:

$$\rho_B(b) \geq 0, \;\; b \in B, \;\; \int\limits_B \rho_B(b)\, db = 1.$$

The ambient temperature participating in (1), (3) does not change in time during heating and its possible values are distributed over the set $\Theta \subset \mathrm{R}$ with a known density $\rho_\Theta(\theta)$ such that

$$\rho_\Theta(\theta) \geq 0, \;\; \theta \in \Theta, \;\; \int\limits_\Theta \rho_\Theta(\theta)\, d\theta = 1.$$

The above problem of optimal control consists in finding the admissible values of the capacities of the sources – $q = q(t) = \left(q^1(t), \dots, q^{N_s}(t)\right) \in Q$ and the coordinates of the sources η, providing the minimum value for the following target functional on average over all the possible values of the initial states of the rod $b \in B$ and the temperature of the external environment $\theta \in \Theta$:

$$\mathcal{J}_T(q,\eta) = \int\limits_B \int\limits_\Theta I_T(q; b, \theta) \rho_\Theta(\theta) \rho_B(b) d\theta\, db, \tag{5}$$

$$I_T(q,\eta; b,\theta) = \int\limits_0^l \chi(x)[u(x,t;q,\eta,b,\theta) - U(x)]^2\, dx + \tag{6}$$

$$+\sigma\|q(t) - \hat{q}(t)\|^2_{L_2^{N_s}[0,T]} + \sigma\|\eta - \hat{\eta}\|^2_{\mathrm{R}^{N_s}}.$$

Here: $u(x,t) = u(x,t;q,\eta,b,\theta)$ is the solution of initial boundary value problem (1)-(3) with the initial condition $u(x,0) = b$, temperature of the external environment θ at admissible values of source capacities $q(t)$; $U(x)$ is the given temperature distribution of the rod, desirable to achieve at the end of the heating process; $\chi(x) \geq 0$, $x \in [0,l]$, is the given weight function; σ, $\hat{q}(t)$ are the given parameters of the regularization of the functional of the problem [10].

Assume now that in N_c given points of the rod $\xi^j \in [\varepsilon, l-\varepsilon]$, $j = 1, \dots, N_c$, temperature values are measured in the neighborhood of these points continuously during its heating:

$$u_j(t) = \int\limits_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(x,t) \nu_\varepsilon\left(x; \xi^j\right) dx\,, \;\; j = 1, \dots, N_c, \;\; t \in [0,T]. \tag{7}$$

The measured values will be used to assign the current values of the capacities of the sources – $q^i(t)$, $i = 1, \dots, N_s$ according to the following linear dependence on the measured temperature values

$$q^i(t) = \sum_{j=1}^{N_c} \alpha_i^j \left[\int\limits_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(x,t)\nu_\varepsilon\left(x;\xi^j\right) dx - \widehat{\omega}_i^j\right], \;\; i = 1, \dots, N_s, \;\; t \in [0,T). \tag{8}$$

where α_i^j, $\widehat{\omega}_i^j$, ξ^j are the feedback parameters, $i = 1, \dots, N_s$, $j = 1, \dots, N_c$ [11], [12].

In (8), the value in square brackets defines the deviation of the temperature measurement at the j-th measurement point from the nominal, relative to the i-th source, value of $\widehat{\omega}_i^j$ at the j-th measurement point. α_i ^ j is the corresponding gain. The nominal values of $\widehat{\omega}_i^j$ are largely determined by the values of the given function $U(x)$ at the measurement points $x = \xi^j$, $j = 1, \dots, N_c$.

Let us write dependencies (8) in the form:

$$q^i(t) = \sum_{j=1}^{N_c} \alpha_i^j \int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(x,t)\nu_\varepsilon(x;\xi^j)\,dx - \sum_{j=1}^{N_c} \alpha_i^j \widehat{\omega}_i^j, \;\; i = 1,\dots,N_s, \;\; t \in [0,T).$$

Denoting

$$\sum_{j=1}^{N_c} \alpha_i^j \widehat{\omega}_i^j = \omega_i, \;\; i = 1,\dots,N_s, \tag{9}$$

dependences (8) take the form:

$$q^i(t) = \sum_{j=1}^{N_c} \alpha_i^j \int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(x,t)\nu_\varepsilon(x;\xi^j)\,dx - \omega_i, \;\; i = 1,\dots,N_s, \;\; t \in [0,T). \tag{10}$$

It is easy to see that, by virtue of (9), the number of independent feedback parameters both when using formulas (8) and (10) is $N_s(N_c+2)$, therefore, for the synthesized controls $q^i(t)$, $i = 1,\dots,N_s$, we will use formula (10).

Note that the continuity with respect to t for $x \in (0,l)$ of the function $u(x,t)$ – the solution of problem (1)-(3) – implies that the capacities $q(t)$ determined from formula (10) with continuous measurements (7) are continuous functions.

Substituting expressions for capacities with continuous feedback (8) into equation (1), we obtain:

$$u_t(x,t) = a^2 u_{xx}(x,t) - \lambda_0[u(x,t)-\theta] + \tag{11}$$

$$+\sum_{i=1}^{N_s} \nu_\varepsilon(x;\eta^i) \int_{t-\Delta t}^{t} \left[\sum_{j=1}^{N_c} \alpha_i^j \int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(\gamma,t)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma - \omega_i \right] \mu_{\Delta t}^i(\tau;t)d\tau,$$

$$x \in (0,l), \quad t \in (0,T].$$

Obviously, equation (11) are integro-differential (loaded) equations studied, for instance, by the author of [13].

By approximating the integral terms in (12), we can obtain pointwise loaded differential equations investigated both in the works of the author of [13] and his students, as well as in [14]-[16].

In general, in the problem considered below, it is required to determine the feedback parameters $\alpha = (\alpha_i^j)$, $\omega = (\omega_i)$, $\eta = (\eta^i)$, $j = 1,\dots,N_c$, $i = 1,\dots,N_s$, taking into account constraints (4) on the capacities of the sources, at which the objective functional will take the minimum possible value. Denote by $y = (\alpha,\omega,\eta) \in \mathrm{R}^N$, $N = N_s(N_c+2)$ the vector of parameters optimized in the problem, which consists of: N_sN_c parameters α_i^j, N_s parameters ω_i and η^i. Objective functional (5), (6) of the problem under consideration is finally written as follows:

$$\mathcal{J}_T(y) = \int_B \int_\Theta I_T(y;b,\theta)\rho_\Theta(\theta)\rho_B(b)d\theta\,db, \tag{12}$$

$$I_T(y;b,\theta) = \int_0^l \chi(x)[u(x,T;y,b,\theta)-U(x)]^2\,dx + \sigma\|y-\hat{y}\|_{\mathrm{R}^N}^2. \tag{13}$$

Here: $u(x,t) = u(x,t;y,b,\theta)$ is the solution of the initial boundary value problem with respect to equation (11) with the optimized parameters $y = (\alpha,\omega,\eta)$, the initial condition $u(x,0) = b$ and ambient temperature θ.

Constraints (4) on the capacities of sources when using dependence (10) for feedback will carry into the following joint integral constraints on the optimized parameters y and temperature at the measurement points ξ^j, $j = 1, \dots, N_c$.

$$\underline{q^i} \le q^i(t;y) = \sum_{j=1}^{N_c} \alpha_i^j \int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(\gamma,t) v_\varepsilon(\gamma;\xi^j)\, d\gamma - \omega_i \le \overline{q^i}, \;\; t \in [0,T), \;\; i = 1, \dots, N_s, \tag{14}$$

which we denote and write in the following equivalent form

$$g^i(t;y) = \left|g_0^i(t;y)\right| - \frac{\overline{q^i} - \underline{q^i}}{2} \le 0, \;\; i = 1, \dots, N_s, \;\; t \in [0,T), \tag{15}$$

$$g_0^i(t;y) = \frac{\overline{q^i} + \underline{q^i}}{2} - q^i(t;y).$$

Considered problem (11), (2)-(4), (12), (13) is a problem of parametric optimal control of a distributed parameter system with feedback with an object, in which a finite-dimensional vector $y \in \mathrm{R}^N$ is optimized. The specific characteristics of the problem are: loading of the differential equation; the value of the objective functional is determined not by one solution to the initial boundary value problem, but by a set of solutions, provided that the initial condition and the ambient temperature take not one value, but a set of values, respectively, from the sets B and Θ; the dimension of the resulting finite-dimensional optimization problem, which is mainly determined by the double product of the number of sources and measurement points, is relatively not very large for problems of synthesis of control of systems with distributed parameters.

3. Approach and formulas for solving the problem

It is easy to show that the functional of the obtained control problem (12), (13) as with continuous (10) is not convex with respect to the optimized feedback parameters y. The admissible range of parameters y, defined by inequalities (14), is also not convex. Nevertheless, the formulas obtained below for the components of the functional gradient can be used to numerically determine the locally optimal values of the feedback parameters or to locally refine any of their values specified by an expert.

For the numerical solution of problem (1)-(6), namely, finding the local minimum of objective functional (12), (13), it is proposed to apply the external penalty method to take into account constraints (15) [10]. Taking into account the possible multiextremity of the problem due to its non-convexity, to solve the problem, we can use different initial points for local optimization of the vector of parameters y and parallelization algorithms for the computational process.

We choose the penalty functional with respect to functional (12), (13) in the following form:

$$\mathcal{J}_{T,\mathcal{R}}(y) = \int_B \int_\Theta I_{T,\mathcal{R}}(y;b,\theta) \rho_\Theta(\theta) \rho_B(b) d\theta\, db, \tag{16}$$

$$I_{T,\mathcal{R}}(y;b,\theta) = \int_0^l \chi(x)[u(x,T;y,b,\theta) - U(x)]^2\, dx + \sigma\|y - \hat{y}\|_{\mathrm{R}^N}^2 + \mathcal{R} G_q(y), \tag{17}$$

$$G_q(y) = \sum_{i=1}^{N_s} \int_0^T \left[g_+^i(t;y)\right]^2 dt.$$

Here $\mathcal{R}$ is the penalty coefficient approaching $+\infty$, the function $g_+^i(t;y) = 0$, if $g^i(t;y) \le 0$, and $g_+^i(t;y) = g^i(t;y)$, if $g^i(t;y) > 0$.

The following theorem takes place.

Theorem. The objective functional $\mathcal{J}_{T,\mathcal{R}}(y)$ of problem (11), (2), (3), (4), (16), (17) for each given penalty coefficient $\mathcal{R}$ and continuous feedback (10) is differentiable with respect to synthesized parameters $y = (\alpha, \omega, \eta)$, and the components of its gradient are determined from the formulas

$$\frac{\partial \mathcal{J}_{T,\mathcal{R}}(y)}{\partial \alpha_i^j} = \int_B \int_\Theta \left\{ -\int_0^T \left(\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon} \psi(x,t) \nu_\varepsilon(\gamma; \eta^i) dx + 2\mathcal{R} g_+^i(t;y) \operatorname{sgn}\left(g_0^i(t;y)\right) \right) \cdot \right.$$
$$\left. \cdot \left(\int_{t-\Delta t}^{t} \mu_{\Delta t}^i(\tau;t) \int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(\gamma,\tau) \nu_\varepsilon(\gamma;\xi^j)\, d\gamma\, d\tau \right) dt + 2\sigma\left(\alpha_i^j - \hat{\alpha}_i^j\right) \right\} \rho_\Theta(\theta) \rho_B(b)\, d\theta\, db, \tag{18}$$

$$\frac{\partial \mathcal{J}_{T,\mathcal{R}}(y)}{\partial \omega_i} = \int_B \int_\Theta \left\{ \int_0^T \left(\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon} \psi(x,t) \nu_\varepsilon(\gamma; \eta^i) dx + 2\mathcal{R} g_+^i(t;y) \operatorname{sgn}\left(g_0^i(t;y)\right) \right) dt + \right.$$
$$\left. + 2\sigma(\omega_i - \hat{\omega}_i) \right\} \rho_\Theta(\theta) \rho_B(b)\, d\theta\, db, \tag{19}$$

$$\frac{\partial \mathcal{J}_{T,\mathcal{R}}(y)}{\partial \eta^i} = \int_B \int_\Theta \left\{ -\int_0^T \left(\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon} \psi_x(x,t) \nu_\varepsilon(x; \eta^i) dx \right) \cdot \right.$$
$$\left. \cdot \left(\int_{t-\Delta t}^{t} \mu_{\Delta t}^i(\tau;t) \left[\sum_{j=1}^{N_c} \alpha_i^j \int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(\gamma,\tau) \nu_\varepsilon(\gamma;\xi^j)\, d\gamma - \omega_i \right] d\tau \right) dt + 2\sigma\left(\eta^i - \hat{\eta}^i\right) \right\} \cdot$$
$$\cdot \rho_\Theta(\theta) \rho_B(b)\, d\theta\, db, \tag{20}$$

where $i = 1, \dots, N_s$, $j = 1, \dots, N_c$, $\psi(x,t) = \psi(x,t;y,b,\theta,\mathcal{R})$ for the current vector of parameters y, permissible under initial conditions $b \in B$, ambient temperature $\theta \in \Theta$ and penalty coefficient $\mathcal{R}$, is a solution to the following conjugate boundary value problem:

$$\psi_t(x,t) = -a^2 \psi_{xx}(x,t) + \lambda_0 \psi(x,t) - \tag{21}$$
$$-\sum_{j=1}^{N_c} \sum_{i=1}^{N_s} \nu_\varepsilon(x;\xi^j) \int_{t-\Delta t}^{t} \mu_{\Delta t}^i(\tau;t) \left\{ \int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon} \alpha_i^j \psi(\gamma,\tau) \nu_\varepsilon(\gamma;\eta^i) d\gamma + 2\mathcal{R} g_+^i(\tau;y) \operatorname{sgn}\left(g_0^i(\tau;y)\right) \right\} d\tau,$$
$$x \in (0,l), \ \ t \in [0,T),$$

$$\psi(x,T) = -2\mu(x)[u(x,T) - U(x)], x \in [0,l], \tag{22}$$

$$\psi_x(0,t) = \lambda \psi(0,t), \ \ \psi_x(l,t) = -\lambda \psi(l,t), \ \ t \in [0,T). \tag{23}$$

Proof. Let us show the differentiability of functional (12), (13) with respect to the vector of parameters y.

Taking into account that the initial states of temperature and the ambient temperature do not depend on the optimized parameters y, then the following holds:

$$\operatorname{grad}_y \mathcal{J}_{T,\mathcal{R}}(y) = \operatorname{grad}_y \int_B \int_\Theta I_{T,\mathcal{R}}(y;b,\theta) \rho_\Theta(\theta) \rho_B(b) d\theta\, db =$$

$$= \int\limits_{B} \int\limits_{\Theta} \text{grad}_y I_{T,\mathcal{R}}(y; b, \theta)\rho_\Theta(\theta)\rho_B(b) d\theta\, db.$$

Therefore, it is sufficient to show the differentiability of functional (13) for any given permissible values of the initial temperature b and the ambient temperature θ.

For the controlling terms in equation (11), we introduce the following notation

$$F(x,t;u,y) = \sum_{i=1}^{N_s} \nu_\varepsilon(x;\eta^i) \int\limits_{t-\Delta t}^{t} \left[\sum_{j=1}^{N_c} \alpha_i^j \int\limits_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\, d\gamma - \omega_i \right] \mu_{\Delta t}^i(\tau;t) d\tau. \quad (25)$$

Then equation (11) can be written as:

$$u_t(x,t) = a^2 u_{xx}(x,t) - \lambda_0[u(x,t) - \theta] + F(x,t;u,y). \quad (26)$$

To investigate the differentiability of functional (13), we use the method of incrementing the independent variables y.

Suppose the vector of parameters $y = (\alpha, \omega, \eta)$ received the increment Δy. The new vector of parameters will be denoted by $y_1 = (y + \Delta y) = (\alpha + \Delta\alpha, \omega + \Delta\omega, \eta + \Delta\eta)$. Then the corresponding increment is obtained for the control action $F(x,t;u,y)$:

$$\Delta F(x,t;u,y) = F(x,t;u,y_1) - F(x,t;u,y). \quad (27)$$

Therefore, the phase variable will also receive an increment

$$\Delta u(x,t;y) = u(x,t;y_1) - u(x,t;y).$$

The function $u(x,t;y_1) = u(x,t;y) + \Delta u(x,t;y)$ here is the solution of initial boundary value problem (11), (2), (3), the function $\Delta u(x,t;y)$, according to (26), is the solution of the problem:

$$\Delta u_t(x,t) = a^2 \Delta u_{xx}(x,t) - \lambda_0 \Delta u(x,t) + \Delta F(x,t;u,y),\ \ x \in (0,l),\ \ t \in (0,T], \quad (28)$$

$$\Delta u(x,0) = 0, \quad x \in [0,l], \quad (29)$$

$$\Delta u_x(0,t) = \lambda \Delta u(0,t),\ \ \Delta u_x(l,t) = -\lambda \Delta u(l,t),\ \ t \in (0,T]. \quad (30)$$

Let us calculate the increment of functional (17):

$$\Delta I_{T,\mathcal{R}}(y;b,\theta) = \Delta I_T(y;b,\theta) + \mathcal{R}\Delta G_q(y). \quad (31)$$

Let us deal with the first term, taking into account (28)–(30):

$$\Delta I_T(y;b,\theta) = I_T(y_1;b,\theta) - I_T(y;b,\theta) = \quad (32)$$

$$= 2\int\limits_0^l \mu(x)[u(x,T) - U(x)]\Delta u(x,T)\, dx + 2\sigma\langle y - \hat{y}, \Delta y\rangle.$$

Moving the right-hand side of equations (28) to the left and multiplying the resulting equality by the as yet arbitrary function $\psi(x,t)$, we integrate both sides with respect to x, $x \in [0,l]$ and with respect to t, $t \in [0,T]$. Adding to (32) the left side of the resulting expression, equal to zero, we obtain

$$\Delta I_T(y;b,\theta) = 2\int\limits_0^l \chi(x)[u(x,T) - U(x)]\Delta u(x,T)\, dx + 2\sigma\langle y - \hat{y}, \Delta y\rangle + \quad (33)$$

$$+ \int\limits_0^T \int\limits_0^l \psi(x,t)\big(\Delta u_t(x,t) - a^2 \Delta u_{xx}(x,t) + \lambda_0 \Delta u(x,t) - \Delta F(x,t;u,y)\big) dx dt.$$

After performing integration by parts in (35), we obtain:

$$\Delta I_T(y;b,\theta) = 2\int_0^l \chi(x)[u(x,T)-U(x)]\Delta u(x,T)\,dx + 2\sigma\langle y-\hat{y},\Delta y\rangle + \tag{34}$$

$$+\int_0^l \psi(x,T)\Delta u(x,T)dx - \int_0^l \psi(x,0)\Delta u(x,0)dx - a^2\int_0^T \psi(l,t)\Delta u_x(l,t)dt +$$

$$+a^2\int_0^T \psi(0,t)\Delta u_x(0,t)dt + a^2\int_0^T \psi_x(l,t)\Delta u(l,t)dt - a^2\int_0^T \psi_x(0,t)\Delta u(0,t)dt -$$

$$-\int_0^T\int_0^l \big(\psi_t(x,t)+a^2\psi_{xx}(x,t)-\lambda_0\psi(x,t)\big)\Delta u(x,t)dxdt - \int_0^T\int_0^l \psi(x,t)\Delta F(x,t;u,y)dxdt.$$

Here $\langle\cdot,\cdot\rangle$ denotes the dot product of vectors.

Let us consider the last term in relation (34). Taking into account (25), (27), we have:

$$\int_0^T\int_0^l \psi(x,t)\Delta F(x,t;u,y)dxdt = \int_0^T\int_0^l \psi(x,t)F(x,t;u,y_1)dxdt - \int_0^T\int_0^l \psi(x,t)\Delta F(x,t;u,y)dxdt =$$

$$= \sum_{i=1}^{N_s}\sum_{j=1}^{N_c}\Delta\alpha_i^j \int_0^T \int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon} \psi(x,t)\nu_\varepsilon(x;\eta^i)dx\left(\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon} u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma\,d\tau\right)dt -$$

$$-\sum_{i=1}^{N_s}\Delta\omega_i\int_0^T\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon}\psi(x,t)\nu_\varepsilon(\gamma;\eta^i)dxdt +$$

$$+\sum_{i=1}^{N_s}\Delta\eta^i\int_0^T\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon}\psi_x(x,t)\nu_\varepsilon(x;\eta^i)dx\left(\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\left[\sum_{j=1}^{N_c}\alpha_i^j\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon}u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma-\omega_i\right]d\tau\right)dt +$$

$$+\sum_{j=1}^{N_c}\sum_{i=1}^{N_s}\int_0^T\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon}\psi(x,t)\nu_\varepsilon(x;\eta^i)dx\left(\alpha_i^j\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon}\Delta u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma\,d\tau\right)dt.$$

Now consider the second term of the functional increment from formula (31):

$$\Delta G_q(y) = G_q(y+\Delta y) - G_q(y) = \sum_{i=1}^{N_s}\int_0^T\left\{\left[g_+^i(t;y_1)\right]^2 - \left[g_+^i(t;y)\right]^2\right\}dt =$$

$$= \sum_{i=1}^{N_s}\int_0^T\left\{\left[\left|g_0^i(t;y_1)\right| - \frac{\overline{q^i}-\underline{q^i}}{2}\right]^2 - \left[\left|g_0^i(t;y)\right| - \frac{\overline{q^i}-\underline{q^i}}{2}\right]^2\right\}dt =$$

$$= -2\sum_{i=1}^{N_s}\sum_{j=1}^{N_c}\Delta\alpha_i^j\int_0^T g_+^i(t;y)\,\mathrm{sgn}\left(g_0^i(t;y)\right)\left(\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon}u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma\,d\tau\right)dt +$$

$$+2\sum_{i=1}^{N_s}\sum_{j=1}^{N_c}\Delta\omega_i\int_0^T g_+^i(t;y)\,\mathrm{sgn}\left(g_0^i(t;y)\right)dt -$$

$$-2\sum_{j=1}^{N_c}\sum_{i=1}^{N_s}\int_0^T g_+^i(t;y)\,\mathrm{sgn}\left(g_0^i(t;y)\right)\left(\alpha_i^j\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon}\Delta u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma\,d\tau\right)dt.$$

It is clear that:

$$\Delta I_{T,\mathcal{R}}(y;b,\theta)=2\int_0^l\chi(x)[u(x,T)-U(x)]\Delta u(x,T)\,dx+2\sigma\langle y-\hat{y},\Delta y\rangle+ \tag{35}$$

$$+\int_0^l\psi(x,T)\Delta u(x,T)dx-\int_0^l\psi(x,0)\Delta u(x,0)dx-a^2\int_0^T\psi(l,t)\Delta u_x(l,t)dt+$$

$$+a^2\int_0^T\psi(0,t)\Delta u_x(0,t)dt+a^2\int_0^T\psi_x(l,t)\Delta u(l,t)dt-a^2\int_0^T\psi_x(0,t)\Delta u(0,t)dt-$$

$$-\int_0^T\int_0^l\left(\psi_t(x,t)+a^2\psi_{xx}(x,t)-\lambda_0\psi(x,t)\right)\Delta u(x,t)dxdt-$$

$$-\sum_{i=1}^{N_s}\sum_{j=1}^{N_c}\Delta\alpha_i^j\left(\int_0^T\left(\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon}\psi(x,t)\nu_\varepsilon(x;\eta^i)dx+2\mathcal{R}g_+^i(t;y)\,\mathrm{sgn}\left(g_0^i(t;y)\right)\right)\cdot\right.$$

$$\left.\cdot\left(\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon}u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma\,d\tau\right)dt+2\sigma\left(\alpha_i^j-\hat{\alpha}_i^j\right)\right)+$$

$$+\sum_{i=1}^{N_s}\Delta\omega_i\left(\int_0^T\left(\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon}\psi(x,t)\nu_\varepsilon(x;\eta^i)dx+2\mathcal{R}g_+^i(t;y)\,\mathrm{sgn}\left(g_0^i(t;y)\right)\right)dt+2\sigma(\omega_i-\hat{\omega}_i)\right)-$$

$$-\sum_{i=1}^{N_s}\Delta\eta^i\int_0^T\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon}\psi_x(x,t)\nu_\varepsilon(x;\eta^i)dx\left(\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\left[\sum_{j=1}^{N_c}\alpha_i^j\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon}u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma-\omega_i\right]d\tau\right)dt+$$

$$-\sum_{j=1}^{N_c}\sum_{i=1}^{N_s}\int_0^T\left(\int_{\eta^i-\varepsilon}^{\eta^i+\varepsilon}\psi(x,t)\nu_\varepsilon(x;\eta^i)dx+2\mathcal{R}g_+^i(t;y)\,\mathrm{sgn}\left(g_0^i(t;y)\right)\right)\cdot$$

$$\cdot\left(\alpha_i^j\int_{t-\Delta t}^{t}\mu_{\Delta t}^i(\tau;t)\int_{\xi^j-\varepsilon}^{\xi^j+\varepsilon}\Delta u(\gamma,\tau)\nu_\varepsilon(\gamma;\xi^j)\,d\gamma\,d\tau\right)dt.$$

Hence, proceeding from the fact that the components of the functional gradient are determined from (35) by the linear parts of its increment with respect to each of the components, it follows that formulas (18)-(20) and conjugate problem (21)-(23) are valid. The theorem is proved.

4. Conclusion

The paper investigates the problem of feedback control of the capacities of point sources during the heating of the rod and optimization of the locations of the sources. To specify the values of the capacities of the sources, a linear dependence on the measured values of the rod temperature at the measurement points has been used. Formulas have been obtained for the components of the gradient in terms of the optimized parameters, which make it possible to use first-order optimization methods for the numerical solution of the problem.

The approach and the scheme for obtaining formulas presented in the paper can be extended to other problems of optimal control of lumped sources in systems with distributed parameters, which are described by other types of initial-boundary value problems, including those with a higher dimension of the spatial variable.

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