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A Krotov-type sufficient optimality condition for stochastic control systems

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ABSTRACT

A Bolza problem described by the nonlinear stochastic Ito differential equation is posed and investigated, and a Krotov-type sufficient optimality condition is established. At the end, it is proved that the Pontryagin maximum principle, which is the most universal necessary condition for optimality, is not only a necessary but also a sufficient optimality condition.

1. Introduction

The optimal control theory has a method for obtaining sufficient conditions for a strong minimum, proposed by V.F. Krotov [1], which was successfully used to solve many applied problems. Further generalization and development of V.F. Krotov's results for various deterministic optimal control problems were the subject of [2-4] and others.

So far, Krotov-type sufficient optimality conditions have been obtained in the problem of optimal control of continuous stochastic systems of various classes with incomplete information about the state vector [5-8], etc.

Note that in the above works, the probability density of a random process, which is a solution to the posed equation, is assumed to be given and satisfies the Fokker-Planck-Kolmogorov equation [9].

In the proposed paper, the solution is sought for all admissible probability densities, and a Krotov-type sufficient optimality condition in stochastic control systems is formulated and proved. Finally, from the sufficient optimality conditions under additional assumptions, a stochastic analogue of the Pontryagin maximum principle is obtained [10-14].

2. Problem statement

Suppose that $(\Omega, \mathrm{F}, \mathrm{P})$ is a full probability space with an allocated non-decreasing flow of σ-algebras F^t, where $F^t = \sigma(w(s),\ t_0 \le s \le t)$, and $w(t)$ is a n-dimensional standard Wiener process. $L_F^2(t_0, t_1;\ R^n)$ is the space of measurable in (t, ω) and F^t – consistent processes, $x(t,\omega): [t_0, t_1]: \Omega \to R^n$, for which

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$$E\int_{t_0}^{t_1}\|x(t)\|^2\,dt<+\infty\,.$$

Here and further, E is the symbol of mathematical expectation.

Suppose we want to minimize the quality functional

$$I(u,x)=E\left\{\varphi\big(x(t_1)\big)+\int_{t_0}^{t_1}f^0(t,x(t),u(t))\,dt\right\}, \tag{1}$$

with constraints

$$\begin{cases}u(t)\in U\subset R^r, t\in T=[t_0,t_1],\\ x(t)\in X(t), t\in T,\end{cases} \tag{2}$$

$$\begin{cases}dx(t)=f\big(t,x(t),u(t)\big)dt+\sigma\big(t,x(t)\big)\,dw(t), \quad t\in T\,,\\ x(t_0)=x_0.\end{cases} \tag{3}$$

Here $x(t)$ is a state vector; $f(t,x,u)(f^0(t,x,u))$ is a given -dimensional vector function (scalar function) continuous in the set of variables; $\sigma(t,x)$: $T\times R^n\to R^{n\times n}$ – $(n\times n)-$ matrix function continuous in the set of variables; $\varphi(x)$ is a given continuously scalar function; $u(t)$ is an r-dimensional vector function of control actions; $X(t)\subset R^n$ is a given non-empty bounded set.

The set of pairs $\big(u(t),x(t)\big)$ possessing the properties listed above and satisfying equations(3) will be called the set of admissible pairs and denoted by D.

The purpose of this study is to find a sufficient optimality condition using a stochastic analogue of the Krotov formalism [1].

3. Auxiliary mathematical constructions and main results

Let us introduce the scalar function $K(t,x)$, which is continuous for all (t,x) and has bounded, continuous partial derivatives $K_t(t,x), K_x(t,x)$, as well as the constructions

$$R(t,x,u)=\frac{\partial K(t,x)}{\partial t}+\frac{\partial K(t,x)}{\partial x}f(t,x,u)-f^0(t,x,u), \tag{4}$$

$$\Phi(x(t_1))=\ \varphi(x(t_1))+K(t_1,x(t_1)).$$

The function $K(t,x)$ possessing the above properties will be called the stochastic analogue of the Krotov function.

The Krotov optimality principle can be formulated as follows.

Theorem 1. (Krotov optimality principle). In order for the admissible pair $(u^0(t),x^0(t))$ on the set of admissible pairs D to be a solution to problem (1)-(3), it suffices that a Krotov function $K(t,x)$ exists such that satisfies the conditions:

$$\begin{cases}\max\limits_{(x(t),v(t))\in D} ER\big(t,x(t),v(t)\big)=ER\big(t,x^0(t),u^0(t)\big),\\ \min\limits_{x(t)\in D_x} E\Phi\big(x(t_1)\big)=E\Phi\big(x^0(t_1)\big),\end{cases} \tag{5}$$

where D_x is a projection of the set D onto the space X [2].

Proof. Suppose that the pair $(u^0(t),x^0(t))$ satisfies the conditions of the theorem. Then the following systems can be written for any other admissible pair $(u(t),x(t))$:

$$\begin{cases} ER\big(t, x^0(t), u^0(t)\big) \geq ER\big(t, x(t), u(t)\big), \\ \qquad \mathrm{E}\Phi\big(x^0(t_1)\big) \leq E\Phi\big(x(t_1)\big). \end{cases}$$

Hence, using the expression for the function R, and integrating both sides over T, and taking into account the properties of the stochastic integral, we have:

$$E\left\{\int_{t_0}^{t_1}\left[\frac{\partial K(t,x^0(t))}{\partial t}+\frac{\partial K(t,x^0(t))}{\partial x}f(t,x^0(t),u^0(t))\right]dt-\int_{t_0}^{t_1}f(t,x^0(t),u^0(t))dt\right\}\geq$$

$$E\left\{\int_{t_0}^{t_1}\left[\frac{\partial K(t,x(t))}{\partial t}+\frac{\partial K(t,x(t))}{\partial x}f(t,x(t),u(t))\right]dt-\int_{t_0}^{t_1}f(t,x(t),u(t))dt\right\}.$$

In other words,

$$E\left\{\int_{t_0}^{t_1}\frac{\partial K(t,x^0(t))}{\partial t}dt-\int_{t_0}^{t_1}f(t,x^0(t),u^0(t))dt\right\}\geq$$

$$E\left\{\int_{t_0}^{t_1}\frac{\partial K(t,x(t))}{\partial t}dt-\int_{t_0}^{t_1}f(t,x(t),u(t))dt\right\}. \tag{6}$$

Further, from inequality (6) we have:

$$E\left\{K(t_1,x^0(t_1))-K(t_0,x^0(t_0))-\int_{t_0}^{t_1}f(t,x^0(t),u^0(t))dt\right\}\geq \tag{7}$$

$$E\left\{K(t_1,x(t_1))-K(t_0,x(t_0))-\int_{t_0}^{t_1}f(t,x(t),u(t))dt\right\}.$$

On the other hand, since the mathematical expectation of the functional $\Phi(x(t_1))$ gets its minimum value along the admissible process $(x^0(t),u^0(t))$, then obtain

$$-E\big\{\varphi\big(x^0(t_1)\big)+K\big(t_1,x^0(t_1)\big)\big\}\geq -E\big\{\varphi\big(x(t_1)\big)+K\big(t_1,x(t_1)\big)\big\}. \tag{8}$$

Summing inequalities (7) and (8), and taking into account that, by assumption $x(t_0)=x^0(t_0)=x_0$, obtain:

$$-E\left\{\varphi\big(x^0(t_1)\big)+\int_{t_0}^{t_1}f\big(t,x^0(t),u^0(t)\big)dt\right\}\geq -E\left\{\varphi\big(x(t_1)\big)+\int_{t_0}^{t_1}f\big(t,x(t),u(t)\big)dt\right\}.$$

Hence, multiplying both sides by (-1), we have

$$I(u^0,x^0)\leq I(u,x),$$

for any pair $(u,x)\in D$.

This means that the process $(u^0(t),x^0(t))$ is the optimal process.

With this, the theorem is fully proved.

4. Sufficient optimality condition in the form of the maximum principle

In this section, it is proved that, under some additional assumptions, the sufficient Krotov condition is the Pontryagin maximum principle for the problem under consideration. In other words, on the basis of sufficient optimality conditions in the stochastic control problem, relations are obtained that allow us to determine the optimal control.

For this purpose, we assume that in considered problem (1)-(3), $X(t)=R^n$, and the functions $f(t,x,u), f^0(t,x,u)$, and $\varphi(x)$ are continuous in the set of variables together with partial derivatives in x.

Then it follows from Theorem 1 that

$$ER\big(t, x^0(t), u^0(t)\big) \geq ER\big(t, x(t), u(t)\big), \tag{9}$$

for all $u(t) \in U, t \in T$, and the necessary extremum conditions [10] (5) are written in the form

$$E\frac{\partial \Phi(x^0(t_1))}{\partial x} = 0, \tag{10}$$

$$E\frac{R\big(t, x^0(t), u^0(t)\big)}{\partial x} = 0, \tag{11}$$

or in expanded form

$$E\left\{\frac{\partial \varphi(x^0(t_1))}{\partial x} + \frac{\partial K(t_1, x^0(t_1))}{\partial x}\right\} = 0, \tag{12}$$

$$E\left\{\frac{\partial^2 K(t, x^0(t))}{\partial t \partial x} + \frac{\partial^2 K\big(t, x^0(t)\big)}{\partial x^2} f(t, x^0(t), u^0(t)) + \frac{\partial K\big(t, x^0(t)\big)}{\partial x} f_x(t, x^0(t), u^0(t) - \right.$$

$$\left. - \frac{\partial f^0\big(t, x^0(t), u^0(t)\big)}{\partial x}\right\} = 0. \tag{13}$$

Suppose that

$$\psi(t) = \frac{\partial K\big(t, x^0(t)\big)}{\partial x},$$

$$H\big(t, x(t), u(t), \psi(t)\big) = \psi'(t) f\big(t, x(t), u(t)\big) - f^0\big(t, x(t), u(t)\big),$$

then from (12)-(13) we have

$$d\psi(t) = -H_x\big(t, x^0(t), u^0(t), \psi^0(t)\big)dt,$$

$$\psi(t_1) = -\frac{\partial \varphi\big(x^0(t_1)\big)}{\partial x}.$$

Further, from (9), taking into account the expressions of the function $R(t, x, u)$ and the Pontryagin function $H\big(t, x(t), u(t), \psi(t)\big)$, we have

$$E\left\{\frac{\partial K(t, x^0(t))}{\partial t} + H\big(t, x^0(t), u^0(t), \psi(t)\big)\right\} \geq E\left\{\frac{\partial K\big(t, x(t)\big)}{\partial t} + H\big(t, x(t), u(t), \psi(t)\big)\right\}.$$

From the last inequality, by similar reasoning from [2], we obtain that the inequality:

$$EH\big(t, x^0(t), u^0(t), \psi(t)\big) \geq EH\big(t, x(t), u(t), \psi(t)\big), \tag{14}$$

holds for all $u(t) \in U, t \in T$.

Inequality (14) shows that the expression $EH\big(t, x(t), u(t), \psi(t)\big)$, for every fixed value of $t \in T$ as a function of u reaches its maximum value at $u^0(t)$, i.e.,

$$\max_{v(t) \in U} EH(t, x(t), u(t), \psi(t)) = EH(t, x^0(t), u^0(t), \psi(t)). \tag{15}$$

In other words, we have established a sufficient optimality condition in the form of the Pontryagin maximum principle [10] for considered control problem (1)-(3) with the above additional assumptions.

Thus, we have proved the following theorem.

Theorem 2. Suppose that $(u^0(t), x^0(t))$ is an admissible process in considered problem (1)-(3). Then, if this process satisfies maximum principle (15), it is optimal.

5. Conclusion

The problem of optimal control for an object described by ordinary stochastic differential Itô equations has been investigated. Krotov-type sufficient optimality conditions have been established, and it has been shown that the maximum principle under additional conditions is not only a necessary but also sufficient optimality condition.

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