Informatics and Control Problems 40 Issue 2 (2020)
journal homepage: www.icp.az

Some probabilistic characteristics of one random process in reliability theory

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ARTICLE INFO

Article history:
Received
Received in revised form
Accepted
Available online

Keywords:
Semi-Markov process
Partial renewal
Renewal rate
Asymptotic methods
Stochastic model

ABSTRACT

A semi-Markov process with discrete intervention of events with one screen is investigated, which describes the behavior of a technical system with partial resource renewal. Asymptotic formulas are found for the mathematical expectation and variance of the first moment of a semi-Markov process entering the zero state.

1. Introduction

Semi-Markov processes with discrete intervention of events are among the most important random processes. In this paper, we will obtain an asymptotic expansion for the mathematical expectation and variance of the first moment of a semi-Markov process entering the zero state. The results obtained are not only of theoretical, but also of significant practical interest. The mathematical expectation of the first moment of a semi-Markov process reaching the zero state means the average time until the first overhaul of a technical system with partial renewal of the resource. Note that various probabilistic characteristics of such random processes were investigated by asymptotic methods in [1-5].

One of the distinctions of this study is that in this stochastic model, the initial state of the process describing the behavior of a technical system is a random variable, and the renewal time and rate are also taken into account.

This work uses the general methods of the theory of random processes, renewal theory and reliability theory.

2. Problem statement

Let a stochastic technical system have a random resource x_0. Assume that the system loses its operability after some random time. After some random time, system renewal starts, which continues for a random amount of time. The renewal rate is constant and equal to $tg\ \alpha$ $(0° < \alpha < 90°)$. It is assumed that the system loses its resources as much as it works.

Our goal is to find an asymptotic formula for the mathematical expectation and variance of

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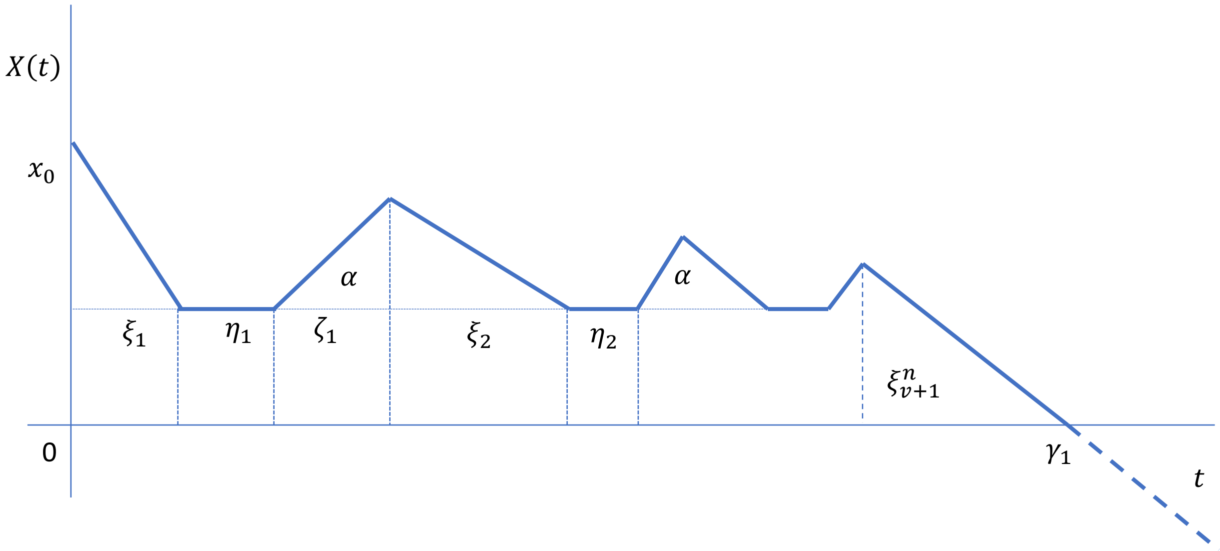
www.icp.az/2020/2-12.pdf

the first passage time of the stochastic process to the zero level.

3. Solution

Suppose that a sequence of independent identically distributed random vectors $\{(\xi_i, \eta_i, \zeta_i)\}_{i=1}^{\infty}$ with independent positive components and a positive random variable x_0 is given on some probability space $(\Omega, \Im, P)$. Here, ξ_k is the duration of the system operation until the k-th failure, η_k is the stopping time until the k-th current repair, ς_k is the duration of the k-th repair.

The functioning of the system looks as follows [6]:



γ_1 – the first moment of the process $X(t)$ entering the zero state – interpreted as a random time before the first passage time of the stochastic system:

$$\gamma_1 = \inf\{t > 0; \mathrm{X}\,(t) = 0\}.$$

It can be seen from the functioning of the system that the random time before the first passage moment γ_1 has the following form:

$$\gamma_1 \; = \sum_{k=0}^{v} (\xi_k + \eta_k + \zeta_k) + \xi_{v+1}\,, \quad \xi_0 = \eta_0 = \varsigma_0 = 0. \;\; X(0) = x_0. \tag{1}$$

Here, v is the the number of renewals before the first passage moment:

$$v = max\left\{n \geq 1; x_0 - \sum_{k=1}^{v} (\xi_k - \zeta_k \cdot tg\ \alpha) \succ 0\right\}.$$

It is taken into account here that $\xi_k - \zeta_k \cdot tg\ \alpha \succ 0$ with probability 1.

Let us formulate the main result of this study in the form of the following theorem.

Theorem. If $\xi_k - \zeta_k \cdot tg\ \alpha \succ 0$ with probability 1, then if $Mx_0 \to \infty$, asymptotic formulas for the mathematical expectation and variance of γ_1 are expressed, respectively, through (9) and (13).

Proof. From the construction of the process [6] describing the stochastic system, it turns out that

$$x_0 - \sum_{k=1}^{v} (\xi_k - \zeta_k \cdot tg\ \alpha) - \xi_{v+1} \leq 0 < x_0 - \sum_{k=1}^{v} (\xi_k - \zeta_k \cdot tg\ \alpha). \tag{2}$$

Inequality (2) can be represented in the following form:

$$x_0 - \xi_{v+1} \leq \sum_{k=1}^{v} (\xi_k - \varsigma_k \cdot tg\ \alpha) < x_0. \tag{3}$$

Proceeding to the mathematical expectation in inequality (3) and using the Wald identity [7], we obtain:

$$Mx_0 - M\xi_1 \leq Mv\ (M\xi_1 - M\zeta_1 \cdot tg\ \alpha) \leq Mx_0. \tag{4}$$

For large values of Mx_0, i.e. if $Mx_0 \to \infty$, then $\frac{M\xi_1}{Mx_0} \to 0$. Then, if $Mx_0 \to \infty$, from (4) it follows that

$$Mv\ (M\xi_1 - M\zeta_1 \cdot tg\ \alpha) \sim\ Mx_0.$$

Hence

$$Mv\ \sim\ \frac{Mx_0}{M\xi_1 - M\zeta_1 \cdot tg\ \alpha}\,. \tag{5}$$

It is clear from the problem statement that $M\xi_1 > tg\ \alpha \cdot M\zeta_1$. Now we can find the mathematical expectation γ_1, i.e. the average time until the first passage moment of the technical system.

Using the Wald identity in (1), we obtain:

$$M\gamma_1 = Mv(M\xi_1 + M\eta_1 + M\zeta_1) + M\xi_1. \tag{6}$$

In (6), taking into account relation (5), we find $M\gamma_1$ in the following way:

$$M\gamma_1 \sim \frac{M\xi_1 + M\eta_1 + M\zeta_1}{M\xi_1 - tg\ \alpha \cdot M\zeta_1} \cdot Mx_0 + M\xi_1. \tag{7}$$

By the refined renewal theorem [8]

$$Mv = \frac{Mx_0}{M\xi_1 - tg\ \alpha \cdot M\zeta_1} + \frac{D\xi_1 + tg^2\alpha D\xi_1 - (M\xi_1 - tg\ \alpha M\zeta_1)^2}{2(M\xi_1 - tg\ \alpha \cdot M\zeta_1)^2} + o(1). \tag{8}$$

Taking into account relation (8) in (6), we find the asymptotic formula for $M\gamma_1$ in the following way:

$$M\gamma_1 = M\xi_1 + (M\xi_1 + M\eta_1 + M\zeta_1) \times$$
$$\times \left(\frac{Mx_0}{M\xi_1 - tg\ \alpha \cdot M\zeta_1} + \frac{D\xi_1 + tg^2\alpha \cdot D\zeta_1 - (M\xi_1 - tg\ \alpha M\zeta_1)^2}{2(M\xi_1 - tg\ \alpha \cdot M\zeta_1)^2} \right) + o(1). \tag{9}$$

Now we calculate the variance from γ_1. Passing to the variance in equality (1), we obtain:

$$D\gamma_1 = D \sum_{k=0}^{v} S_k + D\xi_{v+1}, \quad S_0 = 0, \quad S_k = \xi_k + \eta_k + \zeta_k \qquad k \geq 1. \tag{10}$$

It is known that [7]

$$D \sum_{i=0}^{v} S_k = Mv \cdot DS_1 + Dv(MS_1)^2,$$

where S_k is independent random variables. Then,

$$DS_{\nu} = M\nu(D\xi_1 + D\eta_1 + D\zeta_1) + D\nu(M\xi_1 + M\eta_1 + M\zeta_1)^2. \quad (11)$$

By the refined renewal theorem [8]

$$D\nu = \frac{(D\xi_1 + tg^2\alpha D\zeta_1)\, Mx_0}{(M\xi_1 - tg\,\alpha M\zeta_1)^3} + \frac{5(D\xi_1 + tg^2\alpha D\zeta_1)^2}{4(M\xi_1 - tg\,\alpha \cdot M\zeta_1)^4} -$$

$$-\frac{2\mu_3}{3(M\xi_1 - tg\,\alpha M\zeta_1)^2} + \frac{1}{12} + o(1), \quad (12)$$

where $\mu_3 = M\xi_1^3 - 3M\xi_1^2 . tg\,\alpha M\zeta_1 + 3M\xi_1 . tg^2\alpha M\zeta_1^2 - tg^3\alpha M\zeta_1^3$

Using (8), (12) and (11), in (10) we obtain:

$$D\gamma_1 = D\xi_1 + (D\xi_1 + D\eta_1 + D\zeta_1) \times$$

$$\times\left(\frac{Mx_0}{M\xi_1 - tg\,\alpha M\zeta_1} + \frac{D\xi_1 + tg^2\alpha D\zeta_1 - (M\xi_1 - tg\,\alpha M\zeta_1)^2}{2(M\xi_1 - tg\,\alpha M\zeta_1)^2}\right) +$$

$$+(M\xi_1 + M\eta_1 + M\zeta_1)^2 \left[\frac{(D\xi_1 + tg^2\alpha D\zeta_1)\, Mx_0}{(M\xi_1 - tg\,\alpha M\zeta_1)^3} + \frac{5(D\xi_1 + tg^2\alpha D\zeta_1)^2}{4(M\xi_1 - tg\,\alpha \cdot M\zeta_1)^4} -\right.$$

$$\left. - \frac{2\mu_3}{3(M\xi_1 - tg\,\alpha M\zeta_1)^2} + \frac{1}{12} \right] + o(1). \quad (13)$$

The theorem is proved.

4. Conclusion

In this paper, asymptotic formulas have been found for the mathematical expectation and variance of the first moment when a semi-Markov process passage the zero state, which describes the behavior of a technical system with partial renewal of the resource.

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