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Numerical solution of the problem of identification of sources of oscillations of a hyperbolic system with nonlocal boundary conditions

Y.R. Ashrafova[a,b*], S.R. Rasulova[a]

[a]Baku State University, Baku, Azerbaijan
[b]Institute of Control Systems of Azerbaijan National Academy of Sciences, Baku, Azerbaijan

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The authors investigate a numerical solution to the inverse problem of identifying locations and capacities of sources in a system of arbitrarily connected rods described by a system of differential equations of hyperbolic type with nonseparated boundary conditions. Due to the long duration of the object's operation, the exact values of the initial conditions are not known, but a set of their possible values is given. The considered inverse problem is reduced to a parametric optimal control problem without initial conditions with nonseparated boundary conditions. Formulas for the gradient of the objective functional are obtained, which allow the use of first-order optimization methods for the solution.

1. Introduction

The considered formulation of the identification problem in the variational formulation coincides with the class of problems of optimal control of objects with distributed parameters [1-5]. Examples of such problems can be easily cited from the field of ecology, geophysics, underground hydrodynamics [6-11]. In this paper, this problem is studied in more detail within the framework of the problem of determining the locations and powers of oscillation sources in a system of arbitrarily connected thin homogeneous rods (strings). The problem is described by a system of a large number (equal to the number of rods) of subsystems consisting of two differential equations with partial derivatives of hyperbolic type with impulse actions at the points of location of oscillation sources. The paper investigates a numerical solution to the problem of determining both the capacities and the locations of these actions.

The peculiarity of the considered formulation of the problem lies in the assumption that, due to the long duration, there is no accurate information about the initial state of the oscillatory process at the time of the start of monitoring, and it is not realistic to quickly measure the state of the process at all points of the connected rods simultaneously [12]. But there is information about the set of possible values of the state of the process for all the rods at some initial time instant, and also from that moment in time there are measurement results at certain points of the connected rods. The next peculiarity of the problem lies in the specificity of the boundary conditions, which are given in the form of nonseparated relations between the states at the ends of adjacent rods, determined by the laws of

*Corresponding author.
E-mail addresses: ashrafova.yegana@gmail.com (Y.R. Ashrafova), seva.yuzbekova@yahoo.com (S.R. Rasulova).

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energy balance and continuity of the oscillatory process.

To solve the considered identification problem, it is proposed to use numerical methods of the first-order optimal control, obtaining formulas for the components of the gradient of the minimized objective functional with respect to the identified parameters.

2. Problem statement

Let us represent the considered system of connected homogeneous thin rods in the form of an oriented graph, consisting of N vertices and M arcs. Each arc (rod) is a rod with parameters of the oscillatory process distributed along the length of the arc and time, the state of the process being described by partial differential equations of hyperbolic type. We will use the following notation: I is the set of vertices; $J=\{(k,s): k,s\in I\}$ is the set of arcs (k,s), with the beginning k and the end s, respectively; $l^{(k,s)}$ is the arc length; $J_i^+=\{(j,i): j\in I_i^+\}$ and $J_i^-=\{(i,j): j\in I_i^-\}$ are the sets of arcs, with end and beginning i, respectively, where i and $I_i^- -$ are the sets of vertices adjacent to $I_i^- -$, consisting of n_i^+ and n_i^- vertices, respectively; $I_i=I_i^+\cup I_i^-$; $n_i=n_i^++n_i^-$, in practical applications, $n_i << N, i\in I$ is the set of input and output vertices, i.e., $i\in I^f$, then one of n_i^+ and n_i^- equal to 1, the other is –0; $J^{observ}\subset J$ is the set of arcs in which measurements of the state of processes are carried out, and $\eta^{ks}\in(0,l^{ks})$ is the points of these measurements; $J^{source}\subset J$ is the set of arcs containing sources concentrated at the points $\xi^{ks}\in(0,l^{ks})$; the continuous functions $q^{ks}(t)$ of the capacity of external or internal sources:

$$q^{ks}(t)\begin{cases}\neq 0, & (k,s)\in J^{source},\\ \equiv 0, & (k,s)\in J\backslash J^{source},\end{cases}\quad t\in[t_0,T].$$

Suppose that the capacities $q^{ks}(t)$ and the locations of the point sources of oscillation ξ^{ks} on the arcs of a given set J^{source} are not known and they need to be determined using information about the state of the processes given below.

Based on the meaning of the problem, we assume that there are natural restrictions on the identified functions and parameters:

$$0\le\xi^{ks}\le l^{ks},\ \ \underline{q}\le q^{ks}(t)\le\overline{q},\quad t\in[t_0,T],(k,s)\in J^{source}, \tag{1}$$

where $\underline{q},\overline{q}$ are set values.

Let the state of the oscillatory process of each rod be described by the hyperbolic equation of the following form:

$$u_{tt}^{ks}(x,t)=u_{xx}^{ks}(x,t)-u_t^{ks}(x,t)+q^{ks}(t)\delta(x-\xi^{ks}),\quad x\in(0,l^{ks}),\ \ t\in[t_0,T], \tag{2}$$

with nonseparated boundary conditions:

$$\sum_{k\in I_i^-}\left[\alpha_{ik}^\nu u^{ik}(0,t)+\beta_{ik}^\nu u_x^{ik}(0,t)\right]+\sum_{k\in I_i^+}\left[\alpha_{ki}^\nu u^{ki}\left(l^{ki},t\right)+\beta_{ki}^\nu u_x^{ki}\left(l^{ik},t\right)\right]=\gamma_i^\nu, \tag{3}$$

$$\nu=\overline{1,n_i},i\in I.$$

Here, $u^{ks}(x,t)$ is the state of the (k,s)-th arc at the point x at time instant t, $(k,s)\in J$, $\delta(\cdot)$ is the Dirac function. The given values are are: $l^{ks}>0$, $\alpha_{ik}^\nu,\beta_{ik}^\nu,\ k\in I_i^-$, $\alpha_{ki}^\nu,\beta_{ki}^\nu,\ k\in I_i^+$; γ_i^ν is the ν-th characteristic of external action at the i-th vertex. Such actions, as a rule, differ from zero only at the input and output vertices of the system.

It is important to note the following characteristic of the process under consideration, which is encountered in many practical applications.

It is known that if process (2) lasts long enough, then due to the presence of friction, determined by the second terms in the right-hand sides of equations (2), the influence of the values of the initial conditions on the states of the processes occurring in the arcs weakens over time [12]. Therefore, with long-term monitoring of the process, there is such $\tau, \tau > t_0$ that at $t > \tau$ the state of the processes is significantly affected only by the values of the boundary conditions in the time interval $[t_0, T]$. The value of τ is determined by the parameters of the process [12]. From a practical point of view, this peculiarity is important for the problem under consideration, since taking measurements of the state for all arcs of the system at all points of the phase variable at any one time instant, as a rule, is technically impossible to implement. In practice, measurements of the state of objects distributed in space are carried out at its ends and/or at some specific inner points of arcs.

Therefore, we will assume that at some initial time instant t_0 the initial conditions for process (2)

$$u^{ks}(x, t_0) = \phi_0^{ks}(x, p), \quad u_t^{ks}(x, t_0) = \phi_1^{ks}(x, p), p \in \mathrm{P}, \tag{4}$$

are not determined exactly, but a set of their possible values U^0 is given, for example, in a parametric form:

$$U^0 = \{(\phi_0^{ks}(x; p), \phi_1^{ks}(x; p)) : (k, s) \in J, \ p \in \mathrm{P} \in R^{\nu}\}. \tag{5}$$

P is the given set of possible parameter values, the functions $\phi_0^{ks}(x, p), \ \phi_1^{ks}(x, p)$ are determined to an accuracy of p, with given density functions $(\rho_{\mathrm{P}}^{\phi_0}(p), \rho_{\mathrm{P}}^{\phi_1}(p))$.

Suppose that the functions and parameters participating in initial-boundary value problem (2)-(5) with nonlocal conditions are such that for given values of the capacities of the sources $q_i(t), t \in [0, T]$ and their locations $\xi_i^{ks} \in [0, l^{ks}]$ it has a unique solution for all initial conditions from U^0. We will require the functions $u^{ks}(x, t)$, which determine the solution of problem (2)-(5) to be:

- continuous with respect to the variables x, t on all arcs, $(k, s) \in J$,
- twice continuously differentiable on the arcs without lumped sources, i.e., for $(k, s) \in J \backslash J^{source}$,
- on the arcs containing lumped sources, the functions $u^{ks}(x, t)$ must be continuously differentiable and almost everywhere twice continuously differentiable with respect to the variables x, t, except for the points $x = \xi^{ks} \in (0, l^{ks})$, $(k, s) \in J^{source}$.

Many studies [13] are devoted to the existence and uniqueness of solutions to systems of hyperbolic equations with nonlocal conditions, but, unfortunately, no constructive conditions have been obtained so far. Therefore, we have to be content with the results of numerical studies based on calculations carried out for each specific practical class of problems.

Suppose that, in addition to conditions (3), we measure the state of the processes at some boundary (input and output vertices of the system) points $\eta^{ks} \in (0, l^{ks})$ of some arcs $(k, s) \in J^{observ}$ of the graph:

$$u^{ks}(\eta^{ks}, t) = V_0^{ks}(t), u_t^{ks}(\eta^{ks}, t) = V_1^{ks}(t). \tag{6}$$

Conditions (6) are additional conditions to (3), and we will use them to form an optimized functional that determines the value of the standard deviation between the calculated values of the process state at the observation points and those measured on average for all possible values of the parameters of the initial conditions p from the set P:

$$\widetilde{\Phi}(q, \xi) = \int_{\mathrm{P}} \Phi(q, \xi; p)\, \rho_{\mathrm{P}}(p) dp,$$

$$\Phi(q, \xi, p) = \sum_{(k,s) \in J^{observ.}} \int_{\tau}^{T} [u^{ks}(\eta^{ks}, t; q, \xi, p) - V_0^{ks}(t)]^2 + [u_t^{ks}(\eta^{ks}, t; q, \xi, p) - V_1^{ks}(t)]^2 dt +$$

$$+\varepsilon_1 \sum_{(k,s)\in J^{ist}} \left\| q^{ks}(t) - q_0^{ks}(t) \right\|^2 + \varepsilon_2 \sum_{(k,s)\in J^{ist}} [\xi^{ks} - \xi_0^{ks}].^2 \qquad (7)$$

Here, $u^{ks}(\eta^{ks}, t; p, q, \xi)$ are the calculated values of the state of the process at the time instant t at the observed points as a result of solving the boundary value problem (2)-(5) for admissible values of the parameter $p \in \mathrm{P}$ in initial conditions $\phi_0^{ks}(x,p)$, $\phi_1^{ks}(x,p)$ and given admissible locations and capacities of sources $(\xi, q(t))$; $[\tau, T]$ is the time interval of the process tracking, the states of which no longer depend on the initial conditions at $t = t_0$; $\xi_0^{ks}, q_0^{ks}(t) \in R^m, \varepsilon_1, \varepsilon_2$ are regularization parameters.

Since the initial conditions determined at the time instant t_0 do not affect the process in the interval $[\tau, T]]$, the exact knowledge of the initial value t_0 and the initial functions $\phi_0^{ks}(x;p), \phi_1^{ks}(x;p), (k,s) \in J$ for the value of functional (7) is not essential.

The total number of equations of form (2) is equal to M, and the total number of boundary conditions of form (3) is equal to $2M$. Note that boundary conditions (3) have an essential peculiarity, which is that they are nonseparated (nonlocal) boundary conditions. We will write them in matrix form more generally:

$$A_1 u(0,t) + A_2 u_x(0,t) + B_1 u(l,t) + B_2 u_x(l,t) = \gamma.$$

The following notation is used here:

$$u(0,t) = (u^{1,1}(0,t), u^{1,M}(0,t), \ldots, u^{M,M}(0,t))^T, u(l,t) = (u^{1,1}(l,t), u^{1,M}(l,t), \ldots, u^{M,M}(l,t))^T,$$

$$\gamma = (\gamma_1^1, \ldots, \gamma_1^{n_1}, \ldots, \gamma_N^{n_N})^T.$$

It is assumed that in the $2M \times M$ matrices A_1, A_2, B_1, B_2 composed of the coefficients $\alpha_{ik}^{\nu}, \beta_{ik}^{\nu}$, $k \in I_i^-$, $\alpha_{ki}^{\nu}, \beta_{ki}^{\nu}$, $k \in I_i^+$, the elements, corresponding to nonexistent edges are also equal to 0. The components of the vector $\gamma \in R^{2M}$ corresponding to the conditions relative to the vertices without internal or external sources are equal to 0.

Suppose that the coefficients in conditions (3) are such that the sparse augmented matrix $\mathrm{A} = [\bar{A}, \bar{B}]$ has rank equal to M, where $\bar{A} = [A_1, A_2], \bar{B} = [B_1, B_2]$.. For simplicity of presentation of the calculations carried out below, without loss of generality, we assume the rank of the augmented matrix $\bar{A}$ is equal to $2M$, i.e., $rang[A_1, A_2] = M$. But if $\bar{A}$is an irreversible matrix, then from the matrix A one can extract an invertible submatrix (minor) $\hat{A}$ with rank equal to M. Having changed the order of the columns, we again denote the augmented matrix by $A = [\hat{A}, \breve{B}]$. Here $\breve{B}$ is a matrix composed of the columns of the augmented matrix $[\hat{A}, \breve{B}]$ not included in the matrix $\hat{A}$.

3. Numerical solution of the problem

To solve the problem, we use numerical optimal control methods based on first-order iterative optimization procedures (e.g., the gradient projection method or conjugate gradients), e.g.,

$$\begin{pmatrix} \xi \\ q(t) \end{pmatrix}^{i+1} = \Pr\left[\begin{pmatrix} \xi \\ q(t) \end{pmatrix}^{i} - \alpha\, grad\, \Phi(\xi^i, q^i))\right],\ i = 0,1,\ldots, \qquad (8)$$

where $\Pr(\cdot)$ is the operator of projection of the vector onto the set, which is determined by constraints (1), $\alpha \geq 0$ is the one-dimensional minimization step.

To carry out procedure (8), we obtain formulas for the gradient of objective functional (7) with respect to the identified parameters $(\xi, q(t))$. Taking into account mutual independence, the components of the gradient vector will be obtained independently of each other. Therefore, we assume that on some given arc $(k,s) \in J^{source}$ of the graph there is an oscillation source with unknown value of the capacity $q^{ks}(t)$ and location $\xi^{ks} \in [0; l^{ks}]$, and for other sources this information is available. Using the method of variation of the optimized parameters, we obtain the formulas $grad_{q^{ks}}\Phi(q,\xi,p)$ and $grad_{\xi^{ks}}\Phi(q,\xi,p)$ for any arbitrary possible initial conditions or the corresponding parameters.

Theorem. The components of the gradient of functional (7) by capacity and locations of lumped sources, provided that the edges containing these sources are specified, are determined from the following formulas:

$$grad_{q^{ks}}\widetilde{\Phi}(\xi,q) = \int_{\mathrm{P}} \sum_{(k,s)\in J^{observ.}} \int_{t_0}^{T} \{\psi^{ks}(\xi^{ks},t;q,\xi,p) + 2\varepsilon(q^{ks}(t) - \tilde{q}^{ks}(t))\}\,\rho_{\mathrm{P}}(p)dp, \quad (9)$$

$$t \in [t_0,T],$$

$$grad_{\xi^{ks}}\widetilde{\Phi}(\xi,q) =$$

$$= \int_{\mathrm{P}} \sum_{(k,s)\in J^{observ.}} \left\{ \int_{t_0}^{T} q^{ks}(t)(\psi^{ks}(x,t;q,\xi,p))'_x\,|_{x=\xi^{ks}} dt + 2\varepsilon_2\left(\xi^{ks} - \hat{\xi}^{ks}\right) \right\} \rho_{\mathrm{P}}(p)dp. \quad (10)$$

where $\psi^{ks}(x,t) = \{\psi^{ks}(x,t;q,\xi,p)\}$, $(k,s)\in J$ is the solution to the following adjoint problem:

$$\psi_{tt}^{ks}(x,t) = \begin{cases} \psi_{xx}^{ks}(x,t) + \psi_t^{ks}(x,t), \;\; x\in[0,l^{ks}], \;\; t_0 \le t < \tau, \;\; \psi^{ks}(\tau,x) = 0, \;\; (k,s)\in J, \\ \psi_{xx}^{ks}(x,t) + 2[u^{ks}(x,t;v) - V_0^{ks}(t)]\delta(x-\eta^{ks}), \;\; \tau \le t \le T, \;\; x\in[0,l^{ks}], \;\; (k,s)\in J^{observ.}, \\ \psi_{xx}^{ks}(x,t), \quad \tau \le t \le T, \quad (k,s)\notin J^{observ.}, \;\; x\in[0,l^{ks}], \end{cases} \quad (11)$$

$$\psi^{ks}(\eta^{ks},t) = 2[u_t^{ks}(\eta^{ks},t;v) - V_1^{ks}(t)], \quad \tau \le t \le T, \quad (k,s)\in J^{observ.}, \quad (12)$$

$$\psi^{ks}(x,T) = 0, \psi_t^{ks}(x,T) = 0, \qquad x\in[0,l^{ks}], \quad (13)$$

$$(\bar{A}^{-1}\bar{B})^T \begin{pmatrix} \psi_x(0,t) \\ -\psi(0,t) \end{pmatrix} + \begin{pmatrix} \psi(l,t) \\ -\psi_x(l,t) \end{pmatrix} = 0. \quad (14)$$

Proof. Taking into account the mutual independence of possible initial conditions in U^0, it is clear that for the components of the functional gradient, the following takes place:

$$grad\;\; \widetilde{\Phi}(q,\xi) = grad \int_{\mathrm{P}} \Phi(q,\xi;p)\,\rho_{\mathrm{P}}(p)dp = \int_{\mathrm{P}} grad\;\Phi(q,\xi;p)\,\rho_{\Gamma}(p)dp.$$

Therefore, it suffices to obtain formulas for $grad_{q^{ks}}\Phi(q,\xi,p)$ and $grad_{\xi^{ks}}\Phi(q,\xi,p)$ for any arbitrary possible initial conditions or the corresponding parameters.

In the (k,s)-th equation of system (2), we introduce the notation:

$$v(x,t) = q^{ks}(t)\delta(x-\xi^{ks}). \quad (15)$$

The function $v(x,t)$ will be called an identifiable action. Let $v = v(x,t)$ and $\tilde{v} = \tilde{v}(x,t)$ be two admissible actions corresponding to $u^{ks}(x,t;v)$ and $\tilde{u}^{ks}(x,t;v)$ are the solutions to initial-boundary value problem (2)-(4) under any specifically chosen possible initial conditions $\left(\phi_0^{ks}(x,p),\phi_1^{ks}(x,p)\right),(k,s)\in J.$ We denote $\Delta v = \Delta v(x,t) = \tilde{v}(x,t) - v(x,t), \Delta u^{ks}(x,t) = \tilde{u}^{ks}(x,t;\tilde{v},p) - u^{ks}(x,t;v,p),$. It follows from (2)-(4) that the function $\Delta u^{ks}(x,t)$ is a solution to the following initial-boundary value problem with nonlocal boundary conditions:

$$\Delta u_{tt}^{ks}(x,t) = \Delta u_{xx}^{ks}(x,t) - \Delta u_t^{ks}(x,t) + \Delta v(x,t), \;\; x\in(0,l^{ks}), t\in[t_0,T], (k,s)\in J^{ist}, \quad (16)$$

$$\Delta u_{tt}^{ks}(x,t) = \Delta u_{xx}^{ks}(x,t) - \Delta u_t^{ks}(x,t), \;\; x\in(0,l^{ks}), t\in[t_0,T], \; (k,s)\in J/J^{ist}, \quad (17)$$

$$\sum_{k\in I_i^+}\left[\alpha_{ki}^{\nu}\Delta u^{ki}(0,t)+\beta_{ki}^{\nu}\Delta u_x^{ki}(0,t)\right]+\sum_{k\in I_i^-}\left[\alpha_{ik}^{\nu}\Delta u^{ik}\left(l^{ik},t\right)+\beta_{ik}^{\nu}\Delta u_x^{ki}\left(l^{ik},t\right)\right]=0, \quad (18)$$

$$\nu=\overline{1,n_i}, i\in I,$$

$$\Delta u^{ks}(x,t_0)=0, \Delta u_t^{ks}(x,t_0)=0, s\in I_k^-,\ k\in I. \quad (19)$$

Relations (18) in general form can be written according to (3) in matrix form:

$$A_1\Delta u(0,t)+A_2\Delta u_x(0,t)+B_1\Delta u(l,t)+B_2\Delta u_x(l,t)=0, \quad (20)$$

Then, according to the assumption made, the rank of the augmented matrix $[A_1\,,A_2\,,B_1\,,B_2]$ is equal to M. Multiplying both sides of (20) by $\bar{A}^{-1}$, we get:

$$\begin{pmatrix}\Delta u(0,t)\\ \Delta u_x(0,t)\end{pmatrix}=\bar{A}^{-1}\bar{B}\begin{pmatrix}\Delta u(l,t)\\ \Delta u_x(l,t)\end{pmatrix}, \quad (21)$$

But if $\bar{A}$ is an irreversible matrix, then (21) can be written in the following form:

$$\Delta\hat{u}=\hat{A}^{-1}\breve{B}\Delta\breve{u},$$

where $\Delta\hat{u}$ denotes the vector composed of the elements of the augmented vector $\Delta u=(\Delta u(0,t),\Delta u_x(0,t),\ \Delta u(l,t),\ \Delta u_x(l,t))^T$ corresponding to the numbers of the columns of the matrix $\hat{A}$, and $\Delta\breve{u}$ denotes the vector composed of the elements of the augmented vector Δu corresponding to the numbers of the columns of the matrix $\breve{B}$. But for simplicity of perception of further statements, we will use formulas (21).

The increment of functional (7), taking into account the notation adopted above, can be written in the form:

$$\Delta\Phi(\upsilon,p)=\Phi(\tilde{\upsilon},p)-\Phi(\upsilon,p)=\sum_{(k,s)\in J^{observ.}}2\int_{\tau}^{T}[u^{ks}(\eta^{ks},t;\upsilon,p)-V_0^{ks}(t)]\Delta u^{ks}(\eta^{ks},t;\upsilon,p)+$$

$$+[u_t^{ks}(\eta^{ks},t;\upsilon,p)-V_1^{ks}(t)]u_t^{ks}(\eta^{ks},t;\upsilon,p)dt+2\int_{t_0}^{T}\int_{0}^{l^{ks}}\upsilon(x,t)\Delta\upsilon(x,t)dxdt+R, \quad (22)$$

$$R=\sum_{(k,s)\in J^{observ.}}\int_{\tau}^{T}\left((\Delta u^{ks}(\eta^{ks},t;p,\upsilon))^2+\left(\Delta u_t^{ks}(\eta^{ks},t;p,\upsilon)\right)^2\right)+\|\Delta\upsilon(x,t)\|^2.$$

Taking into account (11)-(13), (16)-(17) in (22) and integrating by parts the integrals over the time and space variables, for the increment of the functional we obtain:

$$\Delta\Phi(\upsilon,p)=\sum_{(k,s)\in J^{observ.}}2\int_{\tau}^{T}\left(\int_{0}^{l^{ks}}[u^{ks}(x,t;\upsilon,p)-V_0^{ks}(t)]\Delta u^{ks}(x,t;\upsilon,p)\delta(x-\eta^{ks})dx+\right.$$

$$\left.+\int_{0}^{l^{ks}}[u_t^{ks}(x,t;\upsilon,p)-V_1^{ks}(t)]\,;\Delta u_t^{ks}(x,t;\upsilon,p)\delta(x-\eta^{ks})dx\right)dt+2\int_{t_0}^{T}\int_{0}^{l^{ks}}\upsilon(x,t)\Delta\upsilon(x,t)dxdt+$$

$$=\sum_{(k,s)\in J^{observ.}}\int_{\tau}^{T}\int_{0}^{l^{ks}}\left[\left(\psi_{xx}^{ks}(x,t)-\psi_{tt}^{ks}(x,t)\right)\Delta u^{ks}(x,t;\upsilon,p)+b^{ks}\psi^{ks}(x,t)\Delta u_t^{ks}(x,t;\upsilon,p)\right]dxdt+$$

$$+R = \sum_{(k,s)\in J} \int_0^{l^{ks}} \Delta u^{ks}(x,t)\, \psi_t^{ks}(x,t)|_{t_0}^T dx - \int_0^{l^{ks}} \psi^{ks}(x,t)\, \Delta u_t^{ks}(x,t)|_{t_0}^T dx +$$

$$+\int_{t_0}^T \psi^{ks}(x,t)\, \Delta u_x^{ks}(x,t)|_0^{l^{ks}} dt - \int_{t_0}^T \Delta u^{ks}(x,t)\, \psi_x^{ks}(x,t)|_0^{l^{ks}} dt -$$

$$-\int_0^{l^{ks}} \psi^{ks}(x,t)\, \Delta u^{ks}(x,t)|_0^{\tau} dt + \int_{t_0}^T \int_0^{l^{ks}} \Delta u^{ks}(x,t) \psi_t^{ks}(x,t)\, dxdt +$$

$$+\int_{\tau}^T \int_0^{l^{ks}} \psi(x,t) \Delta u_t^{ks}(x,t)\, dxdt + \sum_{(k,s)\in J^{source}} \int_{t_0}^T \int_0^{l^{ks}} \psi^{ks}(x,t)\, \Delta v(x,t) dxdt +$$

$$+2\int_{t_0}^T \int_0^{l^{ks}} v(x,t) \Delta v(x,t) dxdt + \sum_{(k,s)\in J} \int_{t_0}^T \int_0^{l^{ks}} \psi^{ks}(x,t) \left(-\Delta u_{tt}^{ks}(x,t) + \Delta u_{xx}^{ks}(x,t)\right.$$

$$\left. - \Delta u_t^{ks}(x,t)\right) dxdt + R = \int_{t_0}^T \psi^{ks}(x,t)\, \Delta u_x^{ks}(x,t)|_0^{l^{ks}} dt -$$

$$-\int_{t_0}^T \Delta u^{ks}(x,t)\, \psi_x^{ks}(x,t)|_0^{l^{ks}} dt + \sum_{(k,s)\in J^{source}} \int_{t_0}^T \int_0^{l^{ks}} \psi^{ks}(x,t)\, \Delta v(x,t) dxdt +$$

$$+2\int_{t_0}^T \int_0^{l^{ks}} v(x,t) \Delta v(x,t) dxdt + R. \quad (23)$$

Using (18)-(21), we get:

$$\sum_{(k,s)\in J} \left(\int_{t_0}^T \psi^{ks}(x,t)\, \Delta u_x^{ks}(x,t)|_0^{l^{ks}} dt - \int_{t_0}^T \Delta u^{ks}(x,t)\, \psi_x^{ks}(x,t)|_0^{l^{ks}} dt \right) =$$

$$= \sum_{(k,s)\in J} \int_{t_0}^T (\psi^{ks}(l^{ks},t) \Delta u_x^{ks}(l^{ks},t) - \psi^{ks}(0,t) \Delta u_x^{ks}(0,t) - \Delta u^{ks}(l^{ks},t) \Delta \psi_x^{ks}(l^{ks},t)$$

$$+ \Delta u^{ks}(0,t) \Delta \psi_x^{ks}(0,t))\, dt = (\psi_x(0,t), -\psi(0,t)) \begin{pmatrix} \Delta u(0,t) \\ \Delta u_x(0,t) \end{pmatrix} +$$

$$+(\psi(l,t), -\psi_x(l,t)) \begin{pmatrix} \Delta u(l,t) \\ \Delta u_x(l,t) \end{pmatrix} = (\psi_x(0,t), -\psi(0,t)) \bar{A}^{-1} \bar{B} \begin{pmatrix} \Delta u(l,t) \\ \Delta u_x(l,t) \end{pmatrix} +$$

$$+(-\psi_x(l,t), \psi(l,t)) \begin{pmatrix} \Delta u(l,t) \\ \Delta u_x(l,t) \end{pmatrix} = (\Delta u(l,t), \Delta u_x(l,t)) \left[(\bar{A}^{-1} \bar{B})^T \begin{pmatrix} \psi_x(0,t) \\ -\psi(0,t) \end{pmatrix} + \begin{pmatrix} \psi(l,t) \\ -\psi_x(l,t) \end{pmatrix} \right].$$

Taking into account (14), using the estimate

$$\int_{t_0}^T \int_0^{l^{ks}} \left(|\Delta u^{ks}(x,t;v)|^2 + |\Delta u_t^{ks}(\eta^{ks},t;v)|^2 \right) dxdt \le C \int_{t_0}^T \int_0^{l^{ks}} |\Delta v(x,t)|^2 dxdt, \quad (k,s) \in J$$

obtained in [14] for the case when the control functions belong to the class of measurable functions $C > 0$) is a constant independent of the value of Δu), for the increment of the functional $\Phi(v)$ we obtain:

$$\Delta\Phi(v,p) = \sum_{(k,s)\in J} \int_{t_0}^{T}\int_{0}^{l^{ks}} \psi^{ks}(x,t;v,p)\,\Delta v dxdt + 2\int_{t_0}^{T}\int_{0}^{l^{ks}} v(x,t)\Delta v dxdt + o(|\Delta v|^2). \tag{24}$$

Due to the increment $\Delta q^{ks}(t)$ of the argument $q^{ks}(t)$, taking into account (9), (19), we have:

$$\Delta\Phi(v,p) = \sum_{(k,s)\in J} \int_{t_0}^{T} \psi^{ks}(\xi^{ks},t;v,p)\,\Delta q^{ks}(t)dxdt + 2\varepsilon \int_{t_0}^{T} \left(q^{ks}(t) - \tilde{q}^{ks}(t)\right)\Delta q^{ks}(t)dt \\ + o(|\Delta q(t)|^2). \tag{25}$$

Taking into account (25), we obtain formula (10) for the components of the gradient of the functional $grad_{q^{ks}}\Phi(q,\xi,p)$ over the value of the capacity $q^{ks}(t)$ at the point$\xi^{ks} \in [0,l^{ks}]$. To obtain the formula for $\xi^{ks} \in [0,l^{ks}]$, in (24)

$$\Delta\Phi(q,\xi,p) = \sum_{(k,s)\in J} \int_{t_0}^{T} q^{ks}(t) \int_{0}^{l^{ks}} [\psi^{ks}(x,t;q,\xi,p)\delta(x-\xi^{ks}+\Delta\xi^{ks}) - \\ -\psi^{ks}(x,t;q,\xi,p)q^{ks}(t)\delta(x-\xi^{ks})]dxdt + 2\varepsilon_2(\xi^{ks}-\hat{\xi}^{ks})\Delta\xi^{ks} + o(\Delta\xi^{ks}) = \\ = \sum_{(k,s)\in J} \int_{t_0}^{T} q^{ks}(t)(\psi^{ks}(x,t;q,\xi,p))'_x|_{x=\xi^{ks}}\Delta\xi^{ks}dt + 2\varepsilon_2(\xi^{ks}-\hat{\xi}^{ks})\Delta\xi^{ks} + o(\Delta\xi^{ks}).$$

Taking this into account, dividing both sides by $\Delta\xi^{ks}$ and proceeding to the limit $\Delta\xi^{ks} \to 0$, we obtain formula (11).

It can be seen from conditions (14) that adjoint problem (11)-(14), like the direct problem, has nonlocal nonseparated boundary conditions, and for its numerical solution, it is necessary to use methods based on special schemes of sweeping of boundary conditions [15–17].

Note. If the numbers of arcs in which unknown sources function are not known in advance, then it is necessary to solve the above problem for various options of groups of arcs with suspected sources. It is clear that the option in which the smallest value of functional (7) is obtained corresponds to the solution of the problem posed. In the case of a large number of arcs with sources or a very large number of arcs M, carrying out such an enumeration on a complex graph is a laborious task. In practice, as a rule, arcs with suspected sources are assigned with sufficient plausibility by experts on the basis of their experience or some other considerations.

It is clear that if, as a result of solving problem (1)-(7) for a sufficiently small value of the functional, we obtain that $|q^{ks}(t)| \le \varepsilon$, $t \in [\tau,T]$, ε is a sufficiently small number, then this means that there are no lumped sources on the (k,s)-th arc as well as in the system on the whole. As a rule, the consequence of the wrong assignment of the arc to determine the source on it in its absence is a large minimal value of the functional or, if the obtained value of ξ^{ks} is such that $0 \le \xi^{ks} \le \varepsilon$ or $l^{ks} - \varepsilon \le \xi^{ks} \le l^{ks}$.

4. Conclusion

The paper investigates the solution to the problem of identifying the locations and capacities of external point sources of oscillations that affect the entire multi-link system as a whole. The state of each thin homogeneous rod (string) is described by a system of hyperbolic differential equations. Based on the results of additional measurements of the state of the process at the ends or at the internal points of the rods, the residual functional has been constructed, for the minimization of which the expression for the gradient has been obtained.

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