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First- and second-order necessary optimality conditions in control problems described by stochastic differential equations

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The authors investigate the problem of optimal control of nonlinear stochastic systems, the mathematical model of which is given by the Ito stochastic differential equation. Under the assumption of openness of the control area by means of the first and second variations of the quality functional, the first- and second-order necessary optimality conditions are derived. In a special case, a stochastic analogue of the Legendre-Clebsch condition is obtained from the necessary second-order optimality condition. Finally, the optimality of controls that are special in the classical sense is investigated and multipoint necessary optimality conditions for singular (in the indicated sense) controls are established.

1. Introduction

The theory of stochastic control systems has many applications in various fields, for example, in the management of technical facilities and information processing in conditions of noises, modeling and evaluation of various economic systems, etc. [1-5].

When describing a stochastic controlled model, a widespread mathematical apparatus is Ito stochastic differential equations [6, 7].

By now, many authors in such control problems have obtained the necessary optimality conditions in the form of the Pontryagin maximum principle, as well as other necessary optimality conditions of various types [8-14].

In [15-18], various types of necessary optimality conditions for control problems for stochastic systems with retarded argument are obtained.

In this paper, using a stochastic analogue of the method proposed in [19-21], under the assumption that the control region is open, a first-order necessary optimality condition (Euler equation) is obtained. Further, using the non-negativity constraint of the second variation of the performance criterion, constructively verifiable second-order necessary optimality conditions are established, including an analogue of the Legendre-Clebsch condition [22, 23]. In addition, special cases (in the classical sense) are studied and a number of second-order necessary optimality conditions are obtained for singular controls.

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2. Problem statement

Suppose (Ω, F, P) is a complete probability space; $w(t)$-dimensional standard Wiener process defined on the complete probable space (Ω, F, P); $L_F^2(t_0, t_1; R^n)$ is the space of random processes measurable with respect to (t, ω), $x(t, \omega)$: $[t_0, t_1]$: $\Omega \to R^n$, for which

$$E\int_{t_0}^{t_1} \|x(t)\|^2 dt < +\infty,$$

where E is the expectation sign.

Assume that the behavior of the control object model is described by the Ito stochastic differential equation

$$dx(t) = f\big(t, x(t), u(t)\big)dt + \sigma\big(t, x(t)\big)dw(t), t \in T, \qquad (1)$$

$$x(t_0) = x_0. \qquad (2)$$

Here, $x(t) \in L_F^2(t_0, t_1; R^n) -$ is the state vector; $f(t, x, u)$ is a given n-dimensional vector function, continuous in the set of variables together with partial derivatives with respect to (x, u) up to the second order inclusive; $\sigma\big(t, x(t)\big): T \times R^n \to R^{n\times n} - (n \times n)$-dimensional matrix function continuous with respect to the set of variables together with partial derivatives with respect to x up to the second order inclusive; $t \in T = [t_0, t_1]$,, moments t_0 and t_1 are given.

$$u(t) \in U_d \equiv \{u(.) \in L_F^2(t_0, t_1; R^r)/u(t) \in U \subset R^r, \text{п. н.}\}, \qquad (3)$$

where U is the given non-empty, bounded and open set; U_d will be called the set of admissible controls.

In what follows, it is assumed that each admissible control $u(t)$, $t \in T$ corresponds to a unique solution to system (1), (2).

The optimal control problem is to minimize the quality functional on the set of admissible controls:

$$S(u) = E\{h(x(t_1))\}. \qquad (4)$$

Here, $h(x)$ is the twice given continuously differentiable scalar function. The admissible control u (t) that affords the minimum to functional (4) under constraints (1)-(2) will be called an optimal control, and the corresponding process (u (t), x (t)) will be called an optimal process.

The admissible control $u(t)$ that affords the minimum to functional (4) under constraints (1)-(2) will be called an optimal control, and the corresponding process $(u(t), x(t))$ will be called an optimal process.

Our goal is to derive a number of necessary optimality conditions in considered problem (1)-(4).

3. First and second variations of the quality functional

Suppose $u(t)$, $t \in T$ is a fixed admissible control. Take one more admissible control $\bar{u}(t) = u(t) + \Delta u(t)$, $t \in T$ and denote the corresponding solutions to problem (1)-(2) by $x(t)$, $\bar{x}(t) = x(t) + \Delta x(t)$.

Let us compose a formula for the increment of the quality functional (4) corresponding to these admissible controls:

$$\Delta S(u) = S(\bar{u}) - S(u) = E\big\{h\big(\bar{x}(t_1)\big) - h\big(x(t_1)\big)\big\}. \qquad (5)$$

On the other hand, it is clear that $\Delta x(t), t \in T$ will satisfy the equation

$$d\Delta x(t) = d[\bar{x}(t) - x(t)] = (f\big(t, \bar{x}(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big))dt +$$

$$+\left(\sigma\left(t,\bar{x}(t)\right)-\sigma\left(t,x(t)\right)\right)dw(t), \tag{6}$$

with the initial condition

$$\Delta x(t_0)=0. \tag{7}$$

Suppose $\psi(t)\in L_F^2(t_0,t_1;R^n)$, $\beta(t)\in L_F^2(t_0,t_1;R^{n\times n})$ is a random process whose stochastic differential has the form:

$$d\psi(t)=\alpha(t)dt+\beta(t)dw(t).$$

Here, $\alpha(t)-$ dimensional measurable and bounded function.

Then, according to the Ito formula [6, 7], we get

$$\begin{gathered}d\left(\psi'(t)\Delta x(t)\right)=d\psi'(t)\Delta x(t)+\psi'(t)d\Delta x(t)+\\+\beta(t)\left[\sigma\left(t,\bar{x}(t)\right)-\sigma\left(t,x(t)\right)\right]dt=d\psi'(t)\Delta x(t)+\\+\psi'(t)\left[\left(f\left(t,\bar{x}(t),\bar{u}(t)\right)-f(t,x(t),u(t))\right)dt+\right.\\\left.+(\sigma(t,\bar{x}(t))-\sigma(t,x(t)))dw(t)\right]+\beta(t)\left[\sigma\left(t,\bar{x}(t)\right)-\sigma\left(t,x(t)\right)\right]dt.\end{gathered} \tag{8}$$

Here and in what follows, everywhere $(')$ is the sign of transposition.

Let us introduce a stochastic analogue of the Hamilton-Pontryagin function as follows:

$$H(t,x,u,\psi)=\psi' f(t,x,u).$$

Now, using the introduced notation, identity (8) can be represented in the form:

$$\begin{gathered}d\left(\psi'(t)\Delta x(t)\right)=d\psi'(t)\Delta x(t)+\\+\left[H\left(t,\bar{x}(t),\bar{u}(t),\psi(t)\right)-H\left(t,x(t),u(t),\psi(t)\right)\right]dt+\\+\psi'(t)\left(\sigma\left(t,\bar{x}(t)\right)-\sigma\left(t,x(t)\right)\right)dw(t)+\beta(t)\left(\sigma\left(t,\bar{x}(t)\right)-\sigma\left(t,x(t)\right)\right)dt.\end{gathered} \tag{9}$$

Let us integrate both sides of identity (9) with respect to T and, taking into account (7), we get that

$$\begin{gathered}\psi'(t_1)\Delta x(t_1)=\int_{t_0}^{t_1}d\psi'(t)\Delta x(t)+\\+\int_{t_0}^{t_1}\left(H\left(t,\bar{x}(t),\bar{u}(t),\psi(t)\right)-H\left(t,x(t),u(t),\psi(t)\right)\right)dt+\\+\int_{t_0}^{t_1}\psi'(t)\left(\sigma\left(t,\bar{x}(t)\right)-\sigma\left(t,x(t)\right)\right)dw(t)+\int_{t_0}^{t_1}\beta(t)\left(\sigma\left(t,\bar{x}(t)\right)-\sigma\left(t,x(t)\right)\right)dt.\end{gathered} \tag{10}$$

Taking into account identity (10) in (5), we will have:

$$\Delta S(u)=E\left\{h\left(\bar{x}(t_1)\right)-h\left(x(t_1)\right)+\psi'(t_1)\Delta x(t_1)-\int_{t_0}^{t_1}d\psi'(t)\Delta x(t)-\right.$$

$$-\int_{t_0}^{t_1}\left(H\left(t,\bar{x}(t),\bar{u}(t),\psi(t)\right)-H\left(t,x(t),u(t),\psi(t)\right)\right)dt-$$

$$-\int_{t_0}^{t_1}\beta(t)(\sigma(t,\bar{x}(t))-\sigma(t,x(t)))\,dt\Bigg\}.$$

To shorten the notation, we denote

$$H_x[t]=H_x(t,x(t),u(t),\psi(t)),H_u[t]=H_u(t,x(t),u(t),\psi(t)),$$

$$H_{xx}[t]=H_{xx}(t,x(t),u(t),\psi(t)),H_{uu}[t]=H_{uu}(t,x(t),u(t),\psi(t)),$$

$$f_x[t]=f_x(t,x(t),u(t)),f_u[t]=f_u(t,x(t),u(t)),$$

$$\sigma_x[t]=\sigma_x(t,x(t)),\sigma_{xx}[t]=\sigma_{xx}(t,x(t)).$$

Further, using the Taylor formula, we obtain:

$$\Delta S(u)=E\Bigg\{h_x'(x(t_1))\Delta x(t_1)+\frac{1}{2}\Delta x'(t_1)h_{xx}(x(t_1))\Delta x(t_1)+\psi'(t_1)\Delta x(t_1)-$$

$$-\int_{t_0}^{t_1}d\psi'(t)\Delta x(t)-\int_{t_0}^{t_1}H_x'[t]\Delta x(t)dt-\int_{t_0}^{t_1}H_u'[t]\Delta u(t)dt-$$

$$-\frac{1}{2}\int_{t_0}^{t_1}\Delta x'(t)H_{xx}[t]\Delta x(t)dt-\frac{1}{2}\int_{t_0}^{t_1}\Delta u'(t)H_{uu}[t]\Delta u(t)dt-$$

$$-\int_{t_0}^{t_1}\Delta x'(t)H_{xu}[t]\Delta u(t)dt-\int_{t_0}^{t_1}\beta(t)\sigma_x[t]\Delta x(t)dt-$$

$$-\frac{1}{2}\int_{t_0}^{t_1}\Delta x'(t)\beta(t)\sigma_{xx}[t]\Delta x(t)dt-\int_{t_0}^{t_1}o_2(\|\Delta z(t)\|^2)dt-$$

$$-\int_{t_0}^{t_1}\beta(t)o_3(\|\Delta x(t)\|^2)dt\Bigg\}. \tag{11}$$

Here, by definition, $\Delta z(t)=(\Delta u(t),\Delta x(t))'$, and the quantities $o_1(\|\Delta x(t_1)\|^2)$, $o_2(\|\Delta z(t)\|^2)$ and $o_3(\|\Delta x(t)\|^2)$are determined, respectively, by the expansion

$$h(\bar{x}(t_1))-h(x(t_1))=h_x'(x(t_1))\Delta x(t_1)+\tfrac{1}{2}\Delta x'(t_1)h_{xx}(x(t_1))\Delta x(t_1)+o_1(\|\Delta x(t_1)\|^2),$$

$$H(t,\bar{x}(t),\bar{u}(t),\psi(t))-H(t,x(t),u(t),\psi(t))=H_x'[t]\Delta x(t)+$$

$$+H_u'[t]\Delta u(t)+\frac{1}{2}\Delta x'(t)H_{xx}[t]\Delta x(t)+\Delta x'(t)H_{xu}[t]\Delta u(t)+$$

$$+\frac{1}{2}\Delta u'(t)H_{uu}[t]\Delta u(t)+o_2(\|\Delta z(t)\|^2),$$

$$\sigma(t,\bar{x}(t))-\sigma(t,x(t))=\sigma_x'[t]\Delta x(t)+\frac{1}{2}\Delta x'(t)\sigma_{xx}[t]\Delta x(t)+o_3(\|\Delta x(t)\|^2).$$

Suppose that the random processes $\psi(t)\in L_F^2(t_0,t_1;R^n)$ and $\beta(t)\in L_F^2(t_0,t_1;R^{n\times n})$ are a solution to the following system of stochastic differential equations (adjoint system):

$$\begin{cases} d\psi(t)=-[H_x[t]+\beta(t)\sigma_x[t]]dt+\beta(t)dw(t), \\ \psi(t_1)=-h_x(x(t_1)). \end{cases} \tag{12}$$

When (12) is fulfilled, increment formula (11) takes the form:

$$\Delta S(u)=E\Bigg\{-\int_{t_0}^{t_1}H_u'[t]\Delta u(t)dt+\frac{1}{2}\Delta x'(t_1)h_{xx}(x(t_1))\Delta x(t_1)-$$

$$-\frac{1}{2}\int_{t_0}^{t_1}\Delta x'(t)\big[H_{xx}[t]+\beta(t)\sigma_{xx}[t]\big]\Delta x(t)dt-\frac{1}{2}\int_{t_0}^{t_1}\Delta u'(t)H_{uu}[t]\Delta u(t)dt-$$
$$-\int_{t_0}^{t_1}\Delta x'(t)H_{xu}[t]\Delta u(t)dt\Bigg\}+\eta(\Delta u). \tag{13}$$

Here by definition

$$\eta(\Delta u)=$$
$$=E\left\{o_1(\|\Delta x(t_1)\|^2)-\int_{t_0}^{t_1}o_2(\|\Delta z(t)\|^2)\,dt-\int_{t_0}^{t_1}\beta(t)o_3(\|\Delta x(t)\|^2)\,dt\right\}. \tag{14}$$

Due to the openness of the control region U, the special increment of the admissible control $u(t)$ can be determined by the formula:

$$\Delta u_\varepsilon(t)=\varepsilon\delta u(t), t\in T. \tag{15}$$

Here, ε is a sufficiently small in absolute value number, $\delta u(t)\in L_F^2(t_0,t_1;R^r)$ is an arbitrary vector function (control variation).

Let $\Delta x_\varepsilon(t)$ denote the special increment of the trajectory $x(t), t\in T$, corresponding to increment (15) of the admissible control $u(t), t\in T$..

From (6), using the Taylor formula, according to the scheme, e.g., in [19], we obtain the validity of the expansion

$$\Delta x_\varepsilon(t)=\varepsilon\delta x(t)+o(\varepsilon;t). \tag{16}$$

Here, $\delta x(t)\in L_F^2(t_0,t_1;R^n)$ (trajectory variation) is the solution to the problem

$$d\delta x(t)=[f_x'[t]\delta x(t)+f_u'[t]\delta u(t)]dt+\sigma_x'[t]\delta x(t)dw(t), \tag{17}$$
$$\delta x(t_0)=0.$$

Following, e.g., [22], we call problem (17) the variational equation for problem (1)-(4).

Note that, based on [24], the solution to problem (17) admits the representation

$$\delta x(t)=\int_{t_0}^{t}R(t,\tau)\delta u(\tau)\,d\tau, \tag{18}$$

where by definition

$$R(t,\tau)=Q(t,\tau)f_u[\tau].$$

Here, the fundamental matrix $Q(t,\tau)$ is the solution to the homogeneous equation:

$$dQ(t,\tau)=f_x'[t]Q(t,\tau)dt+\sigma_x'[t]Q(t,\tau)dw(t),$$
$$Q(t,t)=I\ (I-\text{identity matrix}).$$

Taking into account (14)-(16) in increment formula (13), it is proved that the first and second (in the classical sense) variations of the quality functional, respectively, have the form:

$$\delta^1 S(u;\delta u)=-E\int_{t_0}^{t_1}H_u'[t]\,\delta u(t)dt, \tag{19}$$

$$\delta^2 S(u;\delta u)=$$
$$=E\Bigg(\delta x'(t_1)h_{xx}\big(x(t_1)\big)\delta x(t_1)-\int_{t_0}^{t_1}\delta x'(t)\big[H_{xx}[t]+\beta(t)\sigma_{xx}[t]\big]\delta x(t)dt-$$
$$-2\int_{t_0}^{t_1}\delta x'(t)H_{xu}[t]\delta u(t)dt-\int_{t_0}^{t_1}\delta u'(t)H_{uu}[t]\,\delta u(t)dt\Bigg). \tag{20}$$

4. First- and second-order necessary optimality conditions

Suppose $\big(u(t), x(t)\big)$ is an optimal process. Then, for all $\delta u(t) \in L_F^2(t_0, t_1; R^r)$, according to the results of the calculus of variations (see, e.g., [22]), the first variation of functional (4) is equal to zero, and the second is non-negative, i.e.

$$\delta^1 S(u; \delta u) = 0,$$

$$\delta^2 S(u; \delta u) \geq 0.$$

Thus, along the optimal process $(u(t), x(t))$, for all $\delta u(t) \in L_F^2(t_0, t_1; R^r)$,

$$E\int_{t_0}^{t_1} H_u'[t]\,\delta u(t) = 0, \tag{21}$$

$$E\Bigg(\delta x'(t_1)h_{xx}\big(x(t_1)\big)\delta x(t_1) - \int_{t_0}^{t_1}\delta x'\,(t)\big[H_{xx}[t] + \beta(t)\sigma_{xx}[t]\big]\delta x(t)dt - $$
$$-2\int_{t_0}^{t_1}\delta x'(t)H_{xu}\,[t]\delta u(t)dt - \int_{t_0}^{t_1}\delta u'(t)H_{uu}[t]\,\delta u(t)dt\Bigg) \geq 0. \tag{22}$$

From (21), according to the scheme, e.g., from [25], we obtain that along the optimal process $\big(u(t), x(t)\big)$ the relation

$$EH_u[\theta] = 0, \text{п. н.} \tag{23}$$

holds for almost all arbitrary Lebesgue points of the control $u(t)$, $\theta \in [t_0, t_1)$.

Let us formulate the result obtained.

Theorem 1. The optimality of the admissible control $u(t)$ in problem (1)-(4) requires that equality (23) hold almost for all $\theta \in [t_0, t_1)$.

Optimality condition (23) is a stochastic analogue of the Euler equation for the problem under consideration and is a first-order necessary optimality condition.

In what follows, the admissible control $u(t)$, satisfying Euler equation (23) will be called the classical extremal in problem (1)-(4) [22].

Let us proceed to the derivation of the second-order necessary optimality conditions. For this purpose, we will use an implicit second-order necessary optimality condition (22).

Using representation (18), following, e.g., [19-21], we can obtain the following identities, which will be used in what follows:

$$\delta x'(t_1)h_{xx}\big(x(t_1)\big)\delta x(t_1) =$$
$$= \int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta u'(\tau)R(t_1,\tau)\,h_{xx}\big(x(t_1)\big)R(t_1,s)\delta u(s)d\tau ds, \tag{24}$$

$$\int_{t_0}^{t_1}\delta x'(t)\big[H_{xx}[t] + \beta(t)\sigma_{xx}[t]\big]\,\delta x(t)dt = \tag{25}$$
$$= \int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta u'(\tau)\left[\int_{\max(\tau,s)}^{t_1} R(t,\tau)\big[H_{xx}[t] + \beta(t)\sigma_{xx}[t]\big]R(t,s)dt\right]\delta u(s)\,dsd\tau,$$

$$\int_{t_0}^{t_1}\delta x'(t)H_{xu}\,[t]\delta u(t)dt = \int_{t_0}^{t_1}\left[\int_{t}^{t_1}\delta u'(\tau)H_{xu}[\tau]R(t,\tau)d\tau\right]\delta u(t)dt. \tag{26}$$

Assuming

$$K(\tau, s) = \tag{27}$$
$$= -R(t_1,\tau)h_{xx}\big(x(t_1)\big)R(t_1,s) + \int_{\max(\tau,s)}^{t_1} R(t,\tau)\big[H_{xx}[t] + \beta(t)\sigma_{xx}[t]\big]R(t,s)dt,$$

and taking into account identities (24)-(27), inequality (22) can be written in a compact form:

$$E\left\{\int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta u'(\tau)K(\tau,s)\delta u(s)\,d\tau ds+\int_{t_0}^{t_1}\delta u'(t)H_{uu}\,[t]\delta u(t)dt+\right.$$
$$\left.+2\int_{t_0}^{t_1}\left[\int_{t}^{t_1}\delta u'(\tau)H_{xu}[\tau]R(\tau,t)\,d\tau\right]\delta u(t)dt\right\}\leq 0. \quad (28)$$

Theorem 2. (Second-order necessary optimality condition) The optimality of the classical extremal $u(t), t\in T$ in problem (1)-(4) requires that equality (28) hold almost for all $\delta u(t)\in L_F^2(t_0,t_1;R^r)$.

Note that the deterministic analogue of the matrix function $K(\tau,s)$ was first introduced in the works of K.B. Mansimova.

As can be seen, condition (28) is a general integral necessary condition for the optimality of a classical extremal. But using various singular control variations, from (28) we can obtain a number of easily verifiable pointwise necessary optimality conditions, in particular, a stochastic analogue of the Legendre-Clebsch condition.

Corollary 1 (stochastic analogue of the Legendre-Clebsch condition). The following relation holds along the optimal control $u(t)$

$$Eu'H_{uu}[\theta]u\leq 0, \qquad \text{p.n.} \quad (29)$$

for all $u\in R^r, \theta\in[t_0,t_1)$.

To prove inequality (29), it is sufficient to determincing $\delta u(t)$ in (28) from the formula

$$\delta u_\varepsilon(t)=\begin{cases} u, t\in[\theta,\theta+\varepsilon),\\ 0, t\notin[\theta,\theta+\varepsilon),\end{cases}$$

where $\varepsilon>0$ is a sufficiently small arbitrary number such that $\theta+\varepsilon<t_1$, $\theta\in[t_0,t_1)$ is an arbitrary Lebesgue point of the control $u(t)$, $u\in R^r$ is an arbitrary vector.

Let us introduce a stochastic analogue of the degeneration case of the Legendre-Clebsch condition.

Definition. If the following relation holds along the classical extremal $u(t), t\in T$

$$Eu'H_{uu}[\theta]u=0, \qquad \text{p.n.}$$

for all$u\in R^r, \theta\in[t_0,t_1)$, then this extremal will be called a singular, in the classical sense, control **[**22**]**.

Second-order necessary optimality condition (28) established earlier allows obtaining a number of necessary optimality conditions for singular, in the classical sense, controls.

Suppose $u(t), t\in T$ is a singular, in the classical sense, control, its special variation is determined from the formula

$$\Delta u_\varepsilon(t)=\sum_{i=1}^{m}\delta u(t,\varepsilon,\theta_i,l_i,v_i). \quad (30)$$

Here, m is an arbitrary natural number; $\varepsilon>0$ is a sufficiently small arbitrary number; $l_i\geq 0$, $i=\overline{1,m}$ are is arbitrary numbers; $v_i\in U$, $\theta\in[t_0,t_1)$, $i=\overline{1,m}$ are Lebesgue point of the control $u(t)$, with $t_0\leq\theta_1\leq\theta_2\leq\cdots\leq\theta_m<t_1$, and $\delta u(t,\varepsilon,\theta_i,l_i,v_i)$ is a needle-shaped control variation:

$$\delta u(t,\varepsilon,\theta_i,l_i,v_i)=\begin{cases} v_i, & t\in[\theta_i,\theta_i+l_i\,\varepsilon),\\ 0, & t\in T\backslash[\theta_i,\theta_i+l_i\,\varepsilon).\end{cases} \quad (31)$$

The summation of needle-shaped variations (31) is determined according to the scheme [26].

Hence, taking into account the above, we get the following statement from inequality (28) after some transformations, due to the optimality of $u(t)$.

Theorem 3. Suppose $u(t)$ is an optimal singular, in the classical sense, control in problem (1)-(4). Then for any natural m, the inequality

$$E\left\{\sum_{i=1}^{m}\sum_{j=1}^{m} l_i\, l_j v_i' K\big(\theta_i,\theta_j\big) v_j + \right.$$
$$\left. + \sum_{i=1}^{m} l_i\, v_i' H_{xu}[\theta_i] \left[l_i R(\theta_i,\theta_i)\, v_i + 2\sum_{j=1}^{i-1} l_j\, R\big(\theta_i,\theta_j\big) v_j \right]\right\} \le 0, \tag{32}$$

holds for all $\theta_i \in [t_0,t_1)$ is an arbitrary Lebesgue point of the control $u(t)$ and $l_i \ge 0$, $v_i \in U$, $i=\overline{1,m}$, $(t_0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_m < t_1)$.

Note that the established optimality condition belongs to the class of multipoint necessary optimality conditions for singular controls, and condition (32) makes it possible to significantly reduce the number of singular, in the classical sense, controls suspected of optimality.

From condition (32), one can obtain a number of necessary optimality conditions that are more convenient for verifying.

For $m=1$ and $m=2$, Theorem 3 has, respectively,

Corollary 2. Suppose $u(t)$ is an optimal singular, in the classical sense, control in problem (1)-(4), then its optimality requires that the inequality

$$A(\theta,v) = Ev'\,\{K\,(\theta,\theta) + H_{xu}[\theta] R(\theta,\theta)\} v \le 0, \tag{33}$$

hold for all $\theta \in [t_0,t_1)$ and $v \in U$.

Corollary 3. If $u(t)$ is an optimal singular, in the classical sense, control in problem (1)-(4), then along the process $\big(u(t),x(t)\big)$, the conditions

$$A(\theta_1,v_1) \le 0, \quad A(\theta_2,v_2) \le 0,$$
$$Ev'\,K\,(\theta_2,\theta_1)\,v \le \sqrt{A(\theta_1,v_1)\cdot A(\theta_2,v_2)}\,,$$

are fulfilled for all $\theta_1,\theta_2 \in [t_0,t_1)$, $\theta_1 \le \theta_2$, and $\forall\; v_1,v_2 \in U$.

At the end, it should be noted that inequality (33) is a stochastic analogue of the R. Gabasov-F.M. Kirillova necessary optimality condition obtained by the method of matrix impulses in [22].

5. Conclusion

Applying a stochastic analogue of the modification of the increment method, the first and second variations of the quality functional in the stochastic optimal control problem have been calculated. With their help, the first- and second-order necessary optimality conditions have been formulated and proved.

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