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Construction of a guaranteed suboptimal solution to an integer programming problem with one constraint according to the coefficients of the constraint condition

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ABSTRACT

The concepts of guaranteed solution and guaranteed suboptimal solution according to the coefficients of constraint condition are given for an integer linear programming problem. Based on an economic interpretation of the problem, a method is developed to find a guaranteed suboptimal solution. An algorithm for this method is written and computational experiments are conducted.

1. Introduction

Consider an integer programming problem below:

$$\sum_{j=1}^{n} c_j x_j \rightarrow \max, \tag{1}$$

$$\sum_{j=1}^{n} a_j x_j \leq b, \tag{2}$$

$$0 \leq x_j \leq d_j, \ (j = \overline{1,n}) \text{ və tamdırlar.} \tag{3}$$

Without breaking the generality, we assume that $c_j > 0$, $a_j > 0$, $d_j > 0$, $(j = \overline{1,n})$ and $b > 0$ are given integers.

Note that problem (1)-(3) is well known in the literature [1-7, etc.] and is called the integer knapsack problem. On the other hand, since problem (1)-(3) belongs to the NP-complete class, there are no polynomial time-complexity methods for finding its optimal solution [8]. However, when the number of unknowns is relatively small, there are various methods to solve this problem, such as

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"Branching and Boundaries", "Dynamic programming" or "Combinator" [1, 3, 4, 5, 7, etc.]. Therefore, various approximate solutions of this problem have been developed [2-5, 7, etc.].

We do not give any new solution to problem (1)-(3) in this paper. Our goal is only to provide an economic interpretation of this problem, on the basis of which we can find a solution that guarantees that the value given to function (1) is greater than the previously specified value. However, to achieve this, the coefficients of the constraint condition must be changed to some extent at prescribed intervals, keeping the coefficients of function (1) and the right-hand side of the constraint condition constant.

It should be noted that the concepts of guaranteed solution and guaranteed suboptimal solution for the coefficients of function (1) and the right-hand side of condition (2) are given separately by the authors in [9-14, etc.], developing certain solution methods.

In this paper, we assume that we develop an algorithm for finding a guaranteed solution by minimally changing the coefficients of the constraint condition without changing the coefficients of function (1) and the right side of condition (2).

It should be noted that the parametric integer knapsack problem bags and the problem we are considering in this paper differ significantly from each other. Thus, in parametric problems it is required to find such variation intervals depending on the parameters of coefficients, that the optimal solution does not change. In the case under consideration in this paper, it is required to change the coefficients of the constraint condition minimally in order to ensure that the value of the objective function in the resulting problem is greater than the predetermined number.

To clarify the nature of problem (1)-(3) under consideration, let us give an economic interpretation of this problem.

Suppose that some enterprise (organization, company, etc.) has to produce n types of products expressed in numbers. For the production of one unit of j-th type $(j = \overline{1,n}\ )$ from these products, $a_j, (j = \overline{1,n}\ )$ amount of cost (funds) should be spent, and in this case $c_j, (j = \overline{1,n}\ )$ amount of income is obtained. Let us also assume that the company has allocated the amount b of total resources (raw materials, funds, etc.) for this work. Naturally, the problem should be formulated as follows: what products should be produced in what quantity, so that the total cost of them does not exceed the allocated amount b, and the income obtained is maximal. Obviously, the mathematical model of the problem is obtained as (1)-(3) by including the unknowns $0 \leq x_j \leq d_j,\ (j = \overline{1,n}\ )$ and $x_j,\ (j = \overline{1,n}\ )$, which satisfies the conditions of completeness. Here $d_j,\ (j = \overline{1,n}\ )$ is the maximum number of j-th product to be produced.

2. Problem statement

Suppose that an optimal solution $X^* = (x_1^*, x_2^*, \dots, x_n^*)$ or a suboptimal (approximate) solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ to problem (1)-(3) has been found by a certain method, and the values of function (1)

$$f^* = \sum_{j=1}^{n} c_j x_j^* \quad \text{or} \quad f^s = \sum_{j=1}^{n} c_j x_j^s$$

corresponding to these solutions have been calculated.

Suppose we want to get a value greater than the known value f^* or f^s of function (1). That is, we want to find a guaranteed solution $X^z = (x_1^z, x_2^z, \dots, x_n^z)$ or a guaranteed suboptimal solution $X^{zs} = (x_1^{zs}, x_2^{zs}, \dots, x_n^{zs})$, such that the conditions $f^z \geq f^* + \Delta^z$ or $f^{zs} \geq f^s + \Delta^{zs}$ are satisfied. Here

$$f^z = \sum_{j=1}^{n} c_j x_j^z \ , f^{zs} = \sum_{j=1}^{n} c_j x_j^{zs} \ , \Delta^z = \left[f^* \frac{p}{100} \right] \text{ və ya } \Delta^{zs} = \left[f^s \frac{p}{100} \right],$$

the number p is the percentage of the increase of the numbers f^* or f^s (income according to the economic nature of the problem) and has to be predetermined, and the designation [z] indicates the integer part of the number z.

To achieve this goal, we need to minimally change the costs $a_j, (j = \overline{1,n})$ in the given intervals $[0; \alpha_j]$, $(j = \overline{1,n})$ while keeping the allocated resource b b and the income c_j, $(j = \overline{1,n})$ constant.

Thus, we obtain the following new mathematical model:

$$\delta_j \to min, \qquad (j = \overline{1,n}) \tag{4}$$

$$\sum_{j=1}^{n} c_j x_j \ \geq \ f^* + \Delta^z, \tag{5}$$

$$\sum_{j=1}^{n} (a_j - \delta_j) x_j \ \leq \ b, \tag{6}$$

$$0 \leq \delta_j \leq \ \alpha_j \ , \ (j = \overline{1,n}) \text{ and are integer,} \tag{7}$$

$$0 \leq x_j \leq \ d_j, \ (j = \overline{1,n}) \text{ and are integer.} \tag{8}$$

Here, $c_j > 0, a_j > 0, \ d_j > 0, \ \alpha_j > 0, (j = \overline{1,n}), \ \ b > 0, \ f^*$ and Δ^z are predetermined integers, x_j and δ_j, $(j = \overline{1,n})$ are unknown quantities.

As we see, after this problem has been solved, the costs $a_j > 0, (j = \overline{1,n})$ have to be minimized to δ_j, $(j = \overline{1,n})$.

First of all, note that problem (4)-(8) is a nonlinear (see condition (6)) and a multivariate programming problem. Naturally, this problem is also in the "difficult to solve" NP-complete class. Therefore, in problem (4)-(8) we obtain the following analogous model by writing f^* instead of f^s, and Δ^z instead of Δ^{zs}.

$$\delta_j \to min, \qquad (j = \overline{1,n}) \tag{9}$$

$$\sum_{j=1}^{n} c_j x_j \ \geq \ f^s + \Delta^{zs}, \tag{10}$$

$$\sum_{j=1}^{n} (a_j - \delta_j) x_j \ \leq \ b, \tag{11}$$

$$0 \leq \delta_j \leq \ \alpha_j \ , \ (j = \overline{1,n}) \text{ and are integer,} \tag{12}$$

$$0 \leq x_j \leq \ d_j, \ (j = \overline{1,n}) \text{ and are integer.} \tag{13}$$

Here, as can be seen from conditions (11) and (12), the conditions $\alpha_j < a_j$, $(j = \overline{1,n})$ must be satisfied. On the other hand, problem (9)-(13) is also a nonlinear (see condition (11)) integer programming problem.

3. Theoretical substantiation of the method

Let us first give the following concepts:

Definition 1: If there is a vector $X = (x_1, x_2, \dots, x_n)$ that satisfies conditions (5)-(8) for each of the specified parameters δ_j, $(j = \overline{1,n})$, then we will call this vector a possible solution to problem (4)-(8).

Definition 2: We will call the solution $X^z = (x_1^z, x_2^z, \dots, x_n^z)$ among the possible solutions to problem (4)-(8), which gives minimal values to the parameters δ_j, $(j = \overline{1,n})$, the guaranteed solution of problem (1)-(3) according to the coefficients of the constraint condition.

Naturally, since finding a guaranteed solution to problem (4)-(8) is associated with serious difficulties (especially when there are hundreds of unknowns), it is of great practical importance to develop methods for finding a guaranteed suboptimal (approximate) solution in real time. Therefore, let us give the following definition.

Definition 3: We will call the solution $X^{zs} = (x_1^{zs}, x_2^{zs}, \dots, x_n^{zs})$ among the possible solutions to problem (9)-(13), which gives minimal values to the parameters δ_j, $(j = \overline{1,n})$, the guaranteed suboptimal solution of problem (1)-(3) according to the coefficients of the constraint condition.

In this paper, we developed a method to find a suboptimal solution to problem (9)-(13). This will be a guaranteed suboptimal solution of the problem (1)-(3) according to the coefficients of the constraint condition.

The essence of the method is as follows:

Note that for problem (1)-(3), the concepts of guaranteed solution according to the number b on the right-hand side of constraint (2) and the coefficients c_j, $(j = \overline{1,n})$ of function (1) were given by the authors in [10- 14], developing appropriate algorithms.

In this paper, in contrast, we assume that the costs a_j, $(j = \overline{1,n})$ in the intervals $[0,\ \alpha_j]$, $(j = \overline{1,n})$ have to be minimized, with the allocated resource b and the values c_j, $(j = \overline{1,n})$ remaining constant. For this purpose, in order to find a guaranteed suboptimal solution to problem (9)-(13) according to the coefficients of the constraint condition, we construct the following solution process: first we find by a known method the suboptimal solution $[0,\ \alpha_j]$, $(j = \overline{1,n})$ to problem (1)-(3), the value

$$f^{s_0} = \sum_{j=1}^{n} c_j x_j^{s_0}$$

corresponding to the function (1) and the number $\Delta^s = \left[f^{s_0} \frac{p}{100}\right]$. Here, the designation $[z]$ represents the integer part of the number z, and the number p is the percentage of the specified increase in the number f^{s_0}. Then we take $f^s = f^{s_0}$ under condition (10) and obtain model (9)-(13). The goal here is to find the minimum values of the unknowns δ_j, $(j = \overline{1,n})$ in the intervals $[0,\ \alpha_j]$, $(j = \overline{1,n})$ so that conditions (10)-(13) are satisfied in problem (9)-(13)

Therefore, we initially take $\delta_j := \alpha_j$, $(j = \overline{1,n})$ and keep it in mind writing $a_j' := a_j$, $(j = \overline{1,n})$. Then take $a_j := a_j' - \delta_j$, $(j = \overline{1,n})$. As a result, current problem (1)-(3) is obtained, in which the suboptimal solution $X^0 = (x_1^0, x_2^0, \dots, x_n^0)$ is found by a certain method and it is not difficult to calculate the number

$$f^0 = \sum_{j=1}^{n} c_j x_j^0 .$$

Obviously, $f^0 > f^s + \Delta^s$, because the coefficients a_j, $(j = \overline{1,n})$ are minimized. In order to minimize the quantities δ_j, $(j = \overline{1,n})$, using the principle of dichotomy, let us define the new current quantities δ_j, $(j = \overline{1,n})$: taking $\gamma_j := 0$, $\beta_j := \delta_j$, $t_j := \gamma_j$, $z_j := \beta_j$, $\delta_j := \left[\frac{\gamma_j + \beta_j}{2}\right]$, $(j = \overline{1,n})$, find the coefficients $a_j := a'_j - \delta_j$, $(j = \overline{1,n})$. The result is a new current problem (1)-(3), in which we find the suboptimal solution $X^1 = (x_1^1, x_2^1, \dots, x_n^1)$ and the value

$$f^1 = \sum_{j=1}^{n} c_j x_j^1$$

of function (1) corresponding to this solution. Two ccases are possible here.

Case I : $f^1 \geq f^s + \Delta^s$. **Case II:** $f^1 < f^s + \Delta^s$

In the first case $f^{zs} := f^1$, $X^{zs} := X^1$ is kept in mind. Then, to minimize the quantities δ_j, $(j = \overline{1,n})$, defining $\gamma_j := t_j$, $\beta_j := \delta_j$, $z_j := \beta_j$, $\bar{\delta}_j := \delta_j$, and $\delta_j := \left[\frac{\gamma_j + \beta_j}{2}\right]$, $(j = \overline{1,n})$, we take $a_j := a'_j - \delta_j$, $(j = \overline{1,n})$.

In the second case, i.e., if $f^1 < f^s + \Delta^s$, defining $\gamma_j := \delta_j$, $\beta_j := z_j$, $t_j := \gamma_j$ və $\delta_j := \left[\frac{\gamma_j + \beta_j}{2}\right]$, $(j = \overline{1,n})$, we calculate $a_j := a'_j - \delta_j$, $(j = \overline{1,n})$.

Obviously, only one of the above two cases can be obtained at each step.

Thus, a new current problem (1)-(3) is obtained. Continuing this process, at a certain k-th step, we find the suboptimal solution $X^k = (x_1^k, x_2^k, \dots, x_n^k)$ and the value

$$f^k = \sum_{j=1}^{n} c_j x_j^k$$

of function (1). Obviously, such a solution process can be continued until the relationship $\beta_j - \gamma_j \leq 1$, $(j = \overline{1,n})$ is satisfied for all numbers j, $(j = \overline{1,n})$. In other words, the dichotomy process gives the same result. It should be taken into account here that at any l-th step of the calculation process, if $f^l \geq f^s + \Delta^s$ $(1 \leq l \leq k)$, $f^{zs} := f^l$, $X^{zs} := X^l$ and $\bar{\delta}_j = \delta_j$, $(j = \overline{1,n})$ have to be memorized.

If at a certain k-th step $\beta_j - \gamma_j \leq 1$, $(j = \overline{1,n})$, then the last solution $X^{zs} = (x_1^{zs}, x_2^{zs}, \dots, x_n^{zs})$ obtained in the first case mentioned above is a guaranteed suboptimal solution, and the value f^{zs} is the sought-for guaranteed value of function (1).

4. Results of computational experiments

In order to clarify the explain of the method proposed above, certain computational experiments were conducted. The coefficients of the solved problems are random integers that satisfy the following conditions

$$0 < c_j \leq 999,\ 0 < a_j \leq 99, \mathrm{b} = \left[0.4 \sum_{\mathrm{j}=1}^{\mathrm{n}} a_j d_j\right], 0 \leq d_j \leq 10, (j = \overline{1,\mathrm{n}}).$$

The results of the experiments are shown in the table below. The suboptimal solution to the solved problems was found by the method given in [7, 14]. Here p is the percentage of the increase in the number f^{s_0}. During the computations, $p = 20\%$ was taken.

Table 1
Results of computational experiments

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$
$f^{s_0} = \sum_{j=1}^{n} c_j x_j^{s_0}$	**283460**	**651958**	**1530684**	**3065123**
$\Delta^s = \left[f^{s_0} \frac{p}{100} \right]$	**56692**	**130391**	**306136**	**613024**
$f^{s_0} + \Delta^s$	**340152**	**782349**	**1836820**	**3678147**
f^{zs}	**357159**	**824074**	**1944583**	**3912629**
$f^{zs} - f^{s_0}$	**73699**	**172116**	**413899**	**847506**
$\theta(c_j)$	25.99%	26.39%	27.04%	27.64%
$\sum_{j=1}^{n} a_j$	**6324**	**12154**	**27538**	**62467**
$\sum_{j=1}^{n} a'_j$	**5962**	**11321**	**25398**	**56863**
$\sum_{j=1}^{n} (a_j - a'_j)$	**362**	**833**	**2140**	**5604**
$\theta(a_j)$	5.72%	6.85%	7.77%	8.97%

The main designations in the table have been given earlier in the text of the paper. However, the quantities $\theta(c_j)$ and $\theta(a_j)$ indicate the average percentage of the increase of the functional and the average percentage of the decrease of the coefficients of the constraint condition, respectively. That is,

$$\theta(c_j) = (f^{zs} - f^{s_0}) / f^{s_0} \cdot 100\%, \quad \theta(a_j) = \sum_{j=1}^{n} (a_j - a'_j) / \sum_{j=1}^{n} a_j \cdot 100\%.$$

5. Conclusion

Based on the data in the table, we can conclude the following.

- According to the guaranteed suboptimal solution, if we took the increase in the initial value of the functional as 20%, the total increase varied between 26% -27%.
- The average percentage of the decrease of the coefficients of the constraint condition was 5.72% -8.97%.
- In all the solved problem, with the increase in the number of unknowns, the percentage of the increase in the guaranteed value of the functional increased, and so did the average percentage of the decrease in the coefficients of the constraint condition.
- These results show once again that the method proposed in the paper reflects the reality, and significant results can be obtained by applying this method to practical problems.

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