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Asymptotic results for a semi-Markov process describing the behavior of some stochastic systems

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ABSTRACT

A semi-Markov process with discrete interference of events with one screen, which describes the behavior of a stochastic system with partial recovery of the resource, is investigated. A semi-Markov process is constructed and asymptotic formulas are found for the mathematical expectation and variance of the first moment when the semi-Markov process enters the zero state.

1. Introduction

The study of stochastic processes arising in many applied problems is one of the developing areas of probability theory. Among many random processes, one of the most important are semi-Markov processes with discrete interference of events.

The study of these classes of random processes is not only of theoretical, but also of significant practical interest due to the fact that they are a mathematical model of many real-life phenomena in technology, economics, medicine and other fields of science.

In this paper, a semi-Markov process with discrete interference of events with one screen, which arises in technology, in medicine, etc., is constructed and asymptotic formulas are obtained for the mathematical expectation and variance of the first moment when a semi-Markov process enters the zero state.

Note that in [1]-[6], various probabilistic characteristics of such random processes were investigated using asymptotic methods.

In this paper, the investigated random process is a mathematical model for technical systems with partial recovery of the resource.

One of the distinctions of this paper is that this stochastic model takes into account the rate of decline, which for technical systems with partial restoration of the resource means the rate of decline of the resource of the technical system.

Another distinction of this paper is that in this stochastic model the initial state of the process is a random variable.

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2. Problem statement

Let a stochastic technical system have a random initial state x_0 (system resource). Let us assume that the resource of the system decreases during the activity and after some random time there is a partial recovery of the system resource, which the level of recovery is also a random variable. And then the behavior of the system continues in the same way until the first moment of completion of its activity. It is assumed that the rate of decline in the system resource is constant and equal to $ctg\alpha$ $(0 < \alpha < 90°)$.

Our goal is to construct a stochastic process describing the stochastic behavior of such types of systems and to find an asymptotic formula for the mathematical expectation and variance of the mean time until the first completion of its activity of a stochastic system, used in practical problems.

3. Solution

3.1. Construction of a random process describing the behavior of a system with partial jumps

Let $\{\xi_i\}$, $\{\eta_i\}$, $(i = 1,2, \dots,)$ sequences of independent, identically distributed positive random variables with distribution functions $F_\xi(x), F_\eta(x)$, respectively, are given on some probability space $(\Omega, \Im, P)$, while the quantities ξ_i, η_i are also independent of each other and a positive random variable x_0.

Using the original sequence of random variables, we determine new random variables.

Suppose that $X(0) = x_0 = \lambda_0,\ \tau_0 = 0,\ x_0 \in R_+,\ 0 < \alpha < 90°$

$$\lambda_1 = \max\{0; x_0 - \xi_1 ctg\,\alpha\} + \eta_1$$

$$\lambda_2 = \max\{0; \lambda_1 - \xi_2 ctg\,\alpha\} + \eta_2$$

$$\vdots$$

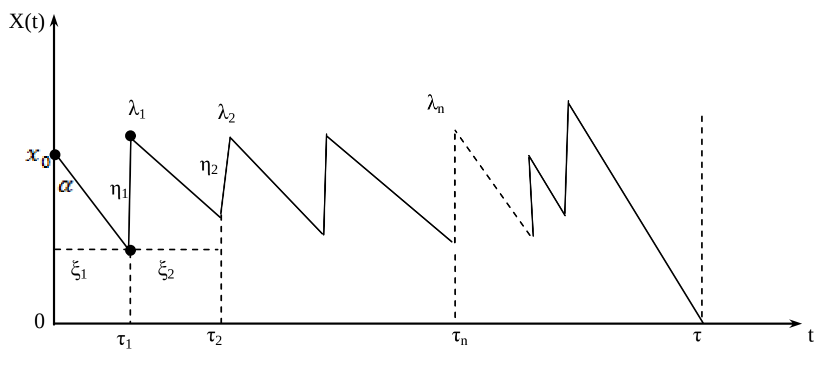
$$\lambda_n = \max\{0; \lambda_{n-1} - \xi_n ctg\,\alpha\} + \eta_n, \qquad n \in N$$

Therefore, $\tau_n = \sum_{k=1}^{n} \xi_k, n \in N$

Now using the above formulas for λ_n and τ_n, the random process $X(t)$ can be represented in general form as follows:

$$X(t) = \lambda_n - (t - \tau_n) ctg\,\alpha, \quad \begin{matrix} \tau_n \le t < \tau_{n+1} \\ n = 0, 1, 2, \dots \end{matrix} \qquad (1)$$

One of the implementations of this process is as follows:



The constructed process is called a semi-Markov process with discrete interference of events with one screen and independent components. Here τ — the first moment when the process $X(t)$ enters the zero state — is interpreted as the time until the first termination of activity of the stochastic system:

$$\tau = \inf\{t > 0; \mathrm{X}\,(t) = 0\}\,.$$

In this paper, the random process $X(t)$, which describes the behavior of a system with partial jumps, is a mathematical model for calculating the residual system resource at time t.

Indeed, if ξ_k is interpreted as the time of the system activity until the -th moment of the system resource recovery, η_k as the growth of the kth system resource recovery, and $ctg\alpha$ as the rate of decline in the system resource, then the random process $X(t)$ means the state of the system at the moment t.

3.2. Asymptotic formula for the mathematical expectation of the first moment when the process $X(t)$ enters the zero state

It is evident from the construction of the process $X(t)$ that

$$\tau = \sum_{k=1}^{\upsilon} \xi_k + \xi_{\upsilon+1}, \tag{2}$$

where τ is the first moment when the process $X(t)$ enters the zero state, i.e.,

$$\tau = \inf\{t > 0; \mathrm{X}\,(t) = 0\}$$

and υ is the number of jumps until the first moment when the process $X(t)$ enters the zero state:

$$\upsilon = max\left\{n \geq 1; x_0 - \sum_{k=1}^{n} \xi_k ctg\alpha + \sum_{k=1}^{n} \eta_k > 0\right\} \tag{3}$$

Here we take into account the fact that $\xi_k ctg\alpha > \eta_k$ with probability 1.

Now we introduce a theorem for finding an asymptotic formula for the mathematical expectation of the first moment when the process $X(t)$ enters the zero state.

First, we introduce the following notation:

$$\mathrm{M}x_0 = \mu \tag{4}$$

Theorem 1. If $\xi_k ctg\alpha > \eta_k$ with probability 1, then for $\mu \to \infty$ the asymptotic formula for the mathematical expectation $\mathrm{M}\tau$ is as follows:

$$\mathrm{M}\tau = \mathrm{M}\xi_1 + \mathrm{M}\xi_1 \left[\frac{\mu}{ctg\alpha\, \mathrm{M}\xi_1 - \mathrm{M}\eta_1} + \frac{ctg^2\alpha D\xi_1 + D\eta_1 - (ctg\alpha\, \mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}{2\,(ctg\alpha\, \mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}\right] + o\,(1)$$

Proof: Using the Wald identity [7] in formula (2) for τ, we find:

$$\mathrm{M}\tau = \mathrm{M}\upsilon \cdot \mathrm{M}\xi_1 + \mathrm{M}\xi_1 \tag{5}$$

Expression (3) can be represented as follows:

$$\upsilon = \max\left\{n \geq 1;\ \sum_{k=1}^{n} (\xi_k ctg\alpha - \eta_k) < x_0\right\} \tag{6}$$

Form the construction of the process describing a stochastic system, it turns out that

$$x_0 - \sum_{k=1}^{\upsilon} (\xi_k ctg\alpha - \eta_k) - ctg\alpha \xi_{\upsilon+1} \leq 0 < x_0 - \sum_{k=1}^{\upsilon} (\xi_k ctg\alpha - \eta_k) \tag{7}$$

Inequality (7) can be represented in the following form:

$$x_0 - ctg\alpha\xi_{v+1} \le \sum_{k=1}^{v} (\xi_k ctg\alpha - \eta_k) < x_0. \tag{8}$$

Proceeding to the mathematical expectation in inequality (7) and using the Wald identity [7], we obtain:

$$Mx_0 - ctg\alpha M\xi_1 \le Mv\,(ctg\alpha M\xi_1 - \mathrm{M}\eta_1) \le Mx_0. \tag{9}$$

Taking into account (4) in (9), we have:

$$\mu - ctg\alpha M\xi_1 \le Mv\,(ctg\alpha M\xi_1 - \mathrm{M}\eta_1) \le \mu \tag{10}$$

For large values of μ, i.e., if $\mu \to \infty$, then $\frac{ctg\alpha M\xi_1}{\mu} \to 0$. Then, for $\mu \to \infty$ it turns out from (10) that

$$Mv\,(ctg\alpha M\xi_1 - \mathrm{M}\eta_1) \to \mu$$

Here, for $\mu \to \infty$ we obtain:

$$\frac{\mathrm{M}v}{\mu} \to \frac{1}{ctg\alpha \mathrm{M}\xi_1 - \mathrm{M}\eta_1} \tag{11}$$

According to the refined renewal theorem for Mv [8] we obtain the following asymptotic expansion:

$$\mathrm{M}v = \frac{\mu}{ctg\alpha\,\mathrm{M}\xi_1 - \mathrm{M}\eta_1} + \frac{ctg^2\alpha D\xi_1 + D\eta_1 - (ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}{2\,(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2} + \mathrm{o}(1) \tag{12}$$

Taking into account this asymptotic expansion for Mv in expression (5), we obtain an asymptotic formula for the mathematical expectation of the first moment of the process $\mathrm{X}(t)$ entering the zero state:

$$\mathrm{M}\tau = \mathrm{M}\xi_1 + \mathrm{M}\xi_1 \left[\frac{\mu}{ctg\alpha\,\mathrm{M}\xi_1 - \mathrm{M}\eta_1} + \frac{ctg^2\alpha D\xi_1 + D\eta_1 - (ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}{2\,(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}\right] + \mathrm{o}(1) \tag{13}$$

The theorem is proved.

3.3. Asymptotic formula for the variance of the first moment when the process $X(t)$ enters the zero state

Theorem 2. If $\xi_k ctg\alpha > \eta_k$ with probability 1, then for $\mu \to \infty$ the asymptotic formula for the variance $D\tau$ is as follows:

$$D\tau = D\xi_1 + D\xi_1 \times$$
$$\times \left(\frac{\mu}{ctg\alpha\,\mathrm{M}\xi_1 - \mathrm{M}\eta_1} + \frac{ctg^2\alpha D\xi_1 + D\eta_1 - (ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}{2\,(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}\right) +$$
$$+(M\xi_1)^2 \left[\frac{(ctg^2\alpha D\xi_1 + D\eta_1)\,\mu}{(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^3} + \frac{5(ctg^2\alpha D\xi_1 + D\eta_1)^2}{4(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^4} - \frac{2\mu_3}{3(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2} + \frac{1}{12}\right] + o(1)$$

Proof: From the construction of the process $X(t)$, it can be seen that the first moment the process $X(t)$ enters the zero state has the following form:

$$\tau = \sum_{k=1}^{v} \xi_k + \xi_{v+1}$$

Hence we find the variance of τ:

$$D\tau = D\sum_{k=1}^{v}\xi_k + D\xi_{v+1}, \tag{14}$$

It is known that [7]

$$D\sum_{k=1}^{v}\xi_k = Mv\cdot D\xi_1 + Dv(M\xi_1)^2, \tag{15}$$

where ξ_k is independent random variables. Then, (14) can be written as follows:

$$D\tau = D\xi_1 + \ Mv\cdot D\xi_1 + Dv(M\xi_1)^2 \tag{16}$$

According to the refined renewal theorem [8]

$$Dv = \frac{(ctg^2\alpha D\xi_1 + D\eta_1)\mu}{(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^3} + \frac{5(ctg^2\alpha D\xi_1 + D\eta_1)^2}{4(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^4} - $$
$$-\frac{2\mu_3}{3(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2} + \frac{1}{12} + o(1)\,, \tag{17}$$

where, $\mu = \mathrm{M}x_0$,

$$\mu_3 = \mathrm{ctg}^3\alpha M\xi_1^3 - 3ctg^2\alpha M\xi_1^2.\mathrm{M}\eta_1 + 3ctg\alpha\ M\xi_1.M\eta_1^2 - M\eta_1^3$$

Using (12) and (17) in (16), we obtain an asymptotic formula for $D\tau$:

$$D\tau = D\xi_1 + D\xi_1 \times$$
$$\times\left(\frac{\mu}{ctg\alpha\ \mathrm{M}\xi_1 - \mathrm{M}\eta_1} + \frac{ctg^2\alpha D\xi_1 + D\eta_1 - (ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}{2\,(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2}\right) +$$
$$+(M\xi_1)^2\left[\ \frac{(ctg^2\alpha D\xi_1 + D\eta_1)\,\mu}{(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^3} + \frac{5(ctg^2\alpha D\xi_1 + D\eta_1)^2}{4(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^4} - \frac{2\mu_3}{3(ctg\alpha\mathrm{M}\xi_1 - \mathrm{M}\eta_1)^2} + \frac{1}{12}\right] + o(1),$$

The theorem is proved.

4. Conclusion

The mathematical expectation and variance of the first moment when the process $\mathrm{X}(t)$ enters the zero state is of practical importance, so, $\mathrm{M}\tau$ means the mean value of the time until the first moment of completion of its activity of the stochastic system, and $D\tau$ means the measure of dispersion of time until the first moment of completion of its activity of the stochastic system from $\mathrm{M}\tau$.

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