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Necessary optimality conditions in one discrete optimal control problem

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ABSTRACT

The article investigates one optimal control problem described by a Volterra-type delay difference equation. A number of necessary optimality conditions are established.

1. Introduction

In [1-6], optimal control problems described by ordinary delay difference equations are studied and a number of necessary optimality conditions are obtained.

In this article, we consider an optimal control problem described by a system of Volterra difference equations with retarded argument. Under some assumptions, first-order necessary optimality conditions are established, such as the Pontryagin maximum principle and its corollaries.

2. Problem statement

Let us consider the problem of the minimum of the terminal functional

$$S(u) = \phi\big(x(t_1)\big), \tag{1}$$

subject to the constraints

$$u(t) \in U \subset R^r, t \in T = \{t_0, t_0 + 1, \dots, t_1 - 1\}, \tag{2}$$

$$x(t+1) = \sum_{\tau=t_0}^{t} f\big(t, \tau, x(\tau), x(\tau - N), u(\tau)\big), t \in T, \tag{3}$$

$$x(t_0 - N) = x_{t_0 - N}, \dots, x(t_0) = x_0. \tag{4}$$

Here, $f(t, \tau, x, y, u)$ is a specified n-dimensional vector function that is continuous in the set of variables together with partial derivatives in (x, y); N is a specified natural number (delay); t_0, t_1 are specified numbers, the difference $t_1 - t_0$ being a natural number; $\phi(x)$ is a specified continuously

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differentiable scalar function; U is a specified non-empty and bounded set; $u(t)$ is a r-dimensional vector of control actions; $x_{t_0-N},\dots,x_0$ are specified constant vectors.

Each control function with the above properties will be called an admissible control. The control function $u(t)$ satisfying the minimum of functional (1) under constraints (2)-(4) will be called an optimal control, and the corresponding process $(u(t),x(t)) -$ and optimal process.

3. Necessary optimality condition in the form of a discrete maximum condition

Assuming that $(u(t),x(t))$ a fixed admissible process, we denote by $(\bar{u}(t)=u(t)+\Delta u(t),$ $\bar{x}(t)=x(t)+\Delta x(t))$ the arbitrary admissible process and write down the increment of the quality functional

$$\Delta S(u)=S(\bar{u})-S(u)=\phi(\bar{x}(t_1))-\phi(x(t)). \qquad (5)$$

It is clear that the increment $\Delta x(t)$ of the "trajectory" $x(t)$ will be the solution to the problem

$$\Delta x(t+1)=\sum_{\tau=t_0}^{t}[f(t,\tau,\bar{x}(\tau),\bar{x}(\tau-N),\bar{u}(\tau))-f(t,\tau,x(\tau),x(\tau-N),u(\tau))], \quad (6)$$

$$\Delta x(t_0-N)=0,\dots,\Delta x(t_0)=0. \qquad (7)$$

Note that problem (6)-(7) (the main initial problem) is an analogue of the Cauchy problem (see, e.g., [3]).

Assume that $\psi(t)$ is as yet unknown n-dimensional discrete vector function. Multiplying both sides of relations (6) from the left scalarly by $\psi(t)$ and then summing up both sides of the resulting relation we will have

$$\sum_{t=t_0}^{t_1-1}\psi'(t)\,\Delta x(t+1)=\sum_{t=t_0}^{t_1-1}[\sum_{\tau=t_0}^{t}\psi'(t)\,[f(t,\tau,\bar{x}(\tau),\bar{x}(\tau-N),\bar{u}(\tau))-$$
$$-f(t,\tau,x(\tau),x(\tau-N),u(\tau))] \qquad (8)$$

Applying to the right-hand side of relation (8) a discrete analogue of the Fubini formula (see, e.g., [2, 7]), formula (8) is written in the form

$$\sum_{t=t_0}^{t_1-1}\psi'(t)\,\Delta x(t+1)=\sum_{t=t_0}^{t_1-1}[\sum_{\tau=t}^{t_1-1}\psi'(\tau)\,[f(\tau,t,\bar{x}(t),\bar{x}(t-N),\bar{u}(t))-$$
$$-f(\tau,t,x(t),x(t-N),u(t))]. \qquad (9)$$

Let us introduce an analogue of the Hamilton-Pontryagin function in the form

$$H\big(t,x(t),x(t-N),u(t),\psi(t)\big)=\sum_{\tau=t}^{t_1-1}\psi'(\tau)\,f(\tau,t,x(t),x(t-N),u(t)). \qquad (10)$$

Taking into account (9), (10), as well as the identity

$$\sum_{t=t_0}^{t_1-1}\psi'(t)\,\Delta x(t+1)=\psi'(t_1-1)x(t_1)+\sum_{t=t_0}^{t_1-1}\psi'(t-1)\,\Delta x(t),$$

the formula for the increment of the quality functional of the problem is written in the form

$$\Delta S(u)=\phi(\bar{x}(t_1)-\phi(x(t))+\psi'(t_1-1)\Delta x(t_1)+\sum_{t=t_0}^{t_1-1}\psi'(t-1)\Delta x(t)-$$

$$-\sum_{t=t_0}^{t_1-1}[H(t,\bar{x}(t),\bar{x}(t-N),\bar{u}(t),\psi(t))-H(t,x(t),x(t-N),u(t),\psi(t))]. \quad (11)$$

Hence, using the Taylor formula, after some transformations of the terms, we have

$$\Delta S(u)=\phi_x'(x(t_1))\Delta x(t_1)+\psi'(t_1-1)\Delta x(t_1)+\sum_{t=t_0}^{t_1-1}\psi'(t-1)\Delta x(t)-$$
$$-\sum_{t=t_0}^{t_1-1}H_x'(t,x(t),x(t-N),u(t),\psi(t))\Delta x(t)-$$
$$-\sum_{t=t_0}^{t_1-1}H_y'(t,x(t),x(t-N),u(t),\psi(t))\Delta x(t-N)-$$
$$-\sum_{t=t_0}^{t_1-1}[H_x'(t,x(t),x(t-N),\bar{u}(t),\psi(t))-H_x(t,x(t),x(t-N),u(t),\psi(t))]'\Delta x(t)-$$
$$-\sum_{t=t_0}^{t_1-1}[H_y'(t,x(t),x(t-N),\bar{u}(t),\psi(t))-H_x(t,x(t),x(t-N),u(t),\psi(t))]'\Delta x(t-N)-$$
$$-\sum_{t=t_0}^{t_1-1}[H(t,x(t),x(t-N),\bar{u}(t),\psi(t))-H(t,x(t),x(t-N),u(t),\psi(t))]+$$
$$+o_1(\|\Delta x_1(t)\|)-\sum_{t=t_0}^{t_1-1}o_2(\|\Delta x(t-N)\|+\|\Delta x(t)\|)\ . \quad (12)$$

Further, in the expression

$$\sum_{t=t_0}^{t_1-1}H_y'(t,x(t),x(t-N),u(t))\Delta x(t-N),$$

substituting $t-N=S$, we get

$$\sum_{t=t_0}^{t_1-1}H_y'(t,x(t),x(t-N),u(t),\psi(t))\Delta x(t-N)=$$
$$=\sum_{t_0-N}^{t_0-1}H_y'((t+N),x(t+N),x(t),u(t+N),\psi(t+N))\Delta x(t)+$$
$$+\sum_{t_0}^{t_1-1-N}H_y'((t+N),x(t+N),x(t),u(t+N),\psi(t+N))\Delta x(t). \quad (13)$$

Suppose that the vector function $\psi(t)$ satisfies the relation

$$\psi(t-1)=H_x(t,x(t),x(t-N),u(t),\psi(t+N))+$$
$$+H_y\big((t+N),x(t+N),x(t),u(t+N),\psi(t+N)\big), \quad (14)$$
$$\psi(t_1)=-\phi_x(x(t_1))$$
$$\psi(t)=0,t>t_1-N \quad (15)$$

System of equations (14) with initial condition (15) will be called the conjugate system in the problem under consideration.

Assume that the set

$$f(t,\tau,x(\tau),x(\tau-N),U)=\{\alpha:\alpha=f(t,\tau,x(\tau),x(\tau-N),u),u\in U\} \quad (16)$$

is convex at all (t,τ).

Then the "disturbance" of the control $u(t)$ can be defined as

$$\Delta u_\varepsilon(t)=v(t,\varepsilon)-u(t) \quad (17)$$

where $\varepsilon\in[0,1]$ is an arbitrary number, and $v(t,\varepsilon)$ is an arbitrary vector such that

$$f(t,\tau,x(\tau),x(\tau-N),v(\tau,\varepsilon))-f(t,\tau,x(\tau),x(\tau-N),u(\tau))=$$
$$=\varepsilon[f(t,\tau,x(\tau),x(\tau-N),v(\tau))-f(t,\tau,x(\tau),x(\tau-N),u(\tau))] \quad (18)$$

where $v(t)\in U,t\in T$ is an arbitrary admissible control.

Through $\Delta x_u(t)$, we determine the special increment of the trajectory$x(t)$ corresponding to the increment $\Delta u_\varepsilon(t)$ of the control $u(t)$.

Using a discrete analogue of the Gronwall-Bellman lemma, we can show that

$$\|\Delta x_\varepsilon(t)\|\le L_1\varepsilon,\ t\in T\cup t_1, \quad (19)$$

where $L_1=const>0$ is some number.

Taking into account estimate (19) and formulas (17), (18), from increment formula (12), we obtain that

$$S(u(t)+\Delta u_\varepsilon(t)-S(u(t))=-\varepsilon\sum_{t=t_0}^{t_1}[H(t,x(t),x(t-N),v(t),\psi(t))-$$
$$-H(t,x(t),x(t-N),u(t),\psi(t))]+o(\varepsilon). \quad (20)$$

Taking into account decomposition (20), we prove.

Theorem 1. If set (16) is convex, then for the optimality of the admissible control $u(t)$ in problem (1)-(4) it is necessary that the inequality

$$\sum_{t=t_0}^{t_1-1}[H(t,x(t),x(t-N),v(t),\psi(t))-H(t,x(t),x(t-N),u(t),\psi(t))]\le 0 \quad (21)$$

hold for all $v(t)\in U,t\in T$.

Inequality (21) is an analogue of the discrete maximum principle (see, e.g., [1-8]) for the problem under consideration.

4. Linearized maximum principle and an analogue of the Euler equation

Suppose that the vector function $f(t,\tau,x,y,u)$ is also continuously differentiable also with respect to u. Then the formula for the increment of quality functional (1) can be represented as

$$\Delta S(u)=-\sum_{t=t_0}^{t_1-1}H'_u\big(t,x(t),x(t-N),u(t),\psi(t)\big)\Delta u(t)+\eta_2(\Delta u), \quad (22)$$

where by definition

$$\eta_2(\Delta u)=o_1(\|\Delta x(t_1)\|)-\sum_{t=t_0}^{t_1-1}o_3(\|\Delta x(t)\|+\|\Delta x(t-N)\|+\|\Delta u(t)\|). \quad (23)$$

Suppose that the set U is convex. Then the special increment of the admissible control $u(t)$ can be defined as

$$\Delta u_{\mu}(t) = \mu[v(t) - u(t)] \tag{24}$$

Here, $\mu \in [0,1]$ is an arbitrary number, $v(t) \in U, t \in T$ is an arbitrary admissible control.

We denote by $\Delta x_{\mu}(t)$ special increment of the trajectory $x(t)$. Using a discrete analogue of the Gronwall-Bellman lemma (see, e.g., [3]), we can prove the correctness of the estimate

$$\|\Delta x_{\mu}(t)\| \leq L_2\mu, t \in T \cup t_1,$$

where $L_2 = const > 0$ is some constant.

Taking into account this estimate, as well as formula (24) from increment (22) of the quality criterion, we obtain the expansion

$$S(u(t) + \Delta u_{\mu}(t) - S(u(t)) = -\mu \sum_{t=t_0}^{t_1-1} H'_u\big(t, x(t), x(t-N), u(t)\big)\big(v(t) - u(t)\big) + o(\mu).$$

It follows from the last expansion, due to the arbitrariness of $\mu \in [0,1]$ that, if $u(t)$ is the optimal control, then

$$\sum_{t=t_0}^{t_1-1} H'_u(t, x(t), x(t-N), u(t), \psi(t))(v(t) - u(t)) \leq 0. \tag{25}$$

Thus, we have proved

Theorem 2. If the set U is convex, then for the optimality of the admissible control $u(t)$ it is necessary that inequality (25) hold for all $v(t) \in U, t \in T$.

Note that inequality (25) is an analogue of the linearized maximum condition (see, e.g., [8, 9]) for the problem under consideration.

Now suppose that the set U is open.

Suppose that ε is a sufficiently small in absolute value number, and $\delta u(t) \in R^r, t \in T$ is an arbitrary r-dimensional discrete and bounded vector function. Due to the openness of the domain of the control U, the admissible control special increment $u(t)$ can be determined from the formula

$$\delta u_{\varepsilon}(t) = \varepsilon \delta u(t). \tag{26}$$

Taking into account (26) using increment formula (22), we prove the correctness of the expansion

$$S(u + \varepsilon\delta u) - S(u) = -\varepsilon \sum_{t=t_0}^{t_1-1} H'_u\big(t, x(t), x(t-N), u(t), \psi(t)\big)\delta u(t) + o(\varepsilon). \tag{27}$$

Expansion (27) implies

Theorem 3. If the set U is open, then for the optimality of the admissible control $u(t)$ it is necessary that relation

$$H_u\big(\theta, x(\theta), x(\theta - N), u(\theta), \psi(\theta)\big) = 0 \tag{28}$$

hold for all $\theta \in T$.

Relation (28) is an analogue of the Euler equation for the problem under consideration.

5. Conclusion

By means of one version of the method of increments, formulas for the first-order increment of the quality functional have been constructed. Investigating them for various special variations of control functions, a number of first-order necessary optimality conditions have been established.

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