Optimality conditions in optimal control problems for a Gursa-Darboux system with a multipoint objective functional

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ABSTRACT

We consider one linear optimal control problem described by a system of hyperbolic equations with the Goursat boundary condition and a multipoint linear objective functional. A necessary and sufficient optimality condition of the Pontryagin maximum principle type is proved.

1. Introduction

There are many studies devoted to various optimal control problems for Goursat-Darboux systems (see, e.g., [1-8]), in which a number of necessary optimality conditions are established under various assumptions.

Beginning with [9-13], optimal control problems described in different time intervals by various ordinary differential equations were studied. Such optimal control problems are called compound optimal control problems or variable-structure optimal control problems.

In this article, one linear optimal control problem is considered, described in different domains by two systems of hyperbolic equations with a multipoint quality functional. A necessary and sufficient optimality condition is proved. The case of a nonlinear convex differentiable objective functional is also investigated. A sufficient optimality condition of the Pontryagin type is proved.

2. Problem statement

Suppose that we have the specified rectangles $D_1 = [t_0, t_1] \times [x_0, x_1], D_2 = [t_1, t_2] \times [x_0, x_1], (t_0 < t_1 < t_2, x_0 < x_1), (T_i, X_i), i = \overline{1,k}$ $(t_0 < T_1 < T_2 < \cdots < T_k \le t_1; x_0 < X_1 < X_2 < \cdots < X_k \le x_1), \theta_i, i = \overline{1,k}$ $(t_1 < \theta_1 < \theta_2 < \cdots < \theta_k \le t_2)$ are specified points; $U_1 \subset R^r, U_2 \subset R^q$ are specified non-empty and bounded sets; $u_i(t,x), i = 1,2$ are measurable and bounded respectively, r- and q-dimensional vector functions satisfying the constraints

$$u_1(t,x) \in U_1 \subset R^r, (t,x) \in D_1,$$
$$u_2(t,x) \in U_2 \subset R^q, (t,x) \in D_2. \quad (1)$$

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The pair $\left(u_1(t,x), u_2(t,x)\right)$ with the above properties will be called admissible control.

Assume that the controlled process is described by systems of linear hyperbolic equations with boundary conditions of the Goursat type, i.e.,

$$\frac{\partial^2 z_1}{\partial t \partial x} = A_1(t,x) z_1 + B_1(t,x) \frac{\partial z_1}{\partial t} + C_1(t,x) \frac{\partial z_1}{\partial x} + f_1(t,x,u_1), \tag{2}$$

$$z_1(t_0, x) = a(x), x \in [x_0, x_1],$$
$$z_1(t, x_0) = b_1(t), t \in [t_0, t_1],$$
$$a(x_0) = b_1(t_0), \tag{3}$$

$$\frac{\partial^2 z_2}{\partial t \partial x} = A_2(t,x) z_2 + B_2(t,x) \frac{\partial z_2}{\partial t} + C_2(t,x) \frac{\partial z_2}{\partial x} + f_2(t,x,u_2), \tag{4}$$

$$z_2(t_1, x) = B(x) z_1(t_1, x), x \in [x_0, x_1],$$
$$z_2(t, x_0) = b_2(t), t \in [t_1, t_2],$$
$$B(x_0) z_1(t_1, x_0) = b_2(t_1). \tag{5}$$

Here, $A_i(t,x), B_i(t,x), C_i(t,x)$ $i = 1, 2$ are specified $(n \times n)$ measurable bounded matrix functions; $f_i(t,x,u_i), i = 1, 2$ are specified n-dimensional vector functions continuous in the set of variables; $a(x), b_i(t), i = 1, 2$ are specified absolutely continuous vector functions; $B(x)$ is a specified continuously differentiable $(n \times n)$ matrix functions.

It is assumed that for a specified admissible control $\left(u_1(t,x), u_2(t,x)\right)$, problem (2)-(5) has a unique absolutely continuous solution $\left(z_1(t,x), z_2(t,x)\right)$ (see, e.g., [3]).

Suppose that c_i, d_i are specified n-dimensional vectors.

Let us consider the problem of finding the minimum value of the multipoint functional

$$S(u_1, u_2) = \sum_{i=1}^{k} c_i{}' z_1(T_i, X_i) + \sum_{i=1}^{k} d_i{}' z_2(\theta_i, X_i) \tag{6}$$

under constraints (1)-(5).

The admissible control $\left(u_1(t,x), u_2(t,x)\right)$ satisfying the minimum value to functional (6) under constraints (1)-(5) will be called an optimal control, and the corresponding process $(u_1(t,x), u_2(t,x), z_1(t,x), z_2(t,x))$ $-$ an optimal process.

3. Formula for the increment of the objective functional and the optimality condition

Assuming that $(u_1(t,x), u_2(t,x), z_1(t,x), z_2(t,x))$ a fixed admissible process, we denote by $\big(\bar{u}_1(t,x) = u_1(t,x) + \Delta u_1(t,x), \bar{u}_2(t,x) = u_2(t,x) + u_2(t,x), \overline{z_1}(t,x) = z_1(t,x) + \Delta z_1(t,x), \overline{z_2}(t,x) = z_2(t,x) + \Delta z_2(t,x)\big)$ the arbitrary admissible process and write down the increment of functional (6).

$$\Delta S(u_1, u_2) = S(\bar{u}_1, \bar{u}_2) - S(u_1, u_2) = \sum_{i=1}^{k} c_i{}' \Delta z_1(T_i, X_i) + \sum_{i=1}^{k} d_i{}' \Delta z_2(\theta_i, X_i). \tag{7}$$

It is clear that the increment $\left(\Delta z_1(t,x), \Delta z_2(t,x)\right)$ of the state $\left(z_1(t,x), z_2(t,x)\right)$ is the solution to the boundary value problem

$$\frac{\partial^2 \Delta z_1(t,x)}{\partial t \partial x} = A_1(t,x) \Delta z_1(t,x) + B_1(t,x) \frac{\partial \Delta z_1(t,x)}{\partial t} +$$
$$+ C_1(t,x) \frac{\partial \Delta z_1(t,x)}{\partial x} + f_1\big(t,x,\bar{u}_1(t,x)\big) - f_1\big(t,x,u_1(t,x)\big), \tag{8}$$

$$z_1(t_0,x)=0, x\in[x_0,x_1],$$
$$z_1(t,x_0)=0, t\in[t_0,t_1], \tag{9}$$
$$\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}=A_2(t,x)\Delta z_2(t,x)+B_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial t}+$$
$$+C_2(t,x)\frac{\partial\Delta z_2(t,x)}{\partial x}+f_2\big(t,x,\bar u_2(t,x)\big)-f_2\big(t,x,u_2(t,x)\big), \tag{10}$$
$$z_2(t_1,x)=B(x)\Delta z_1(t_1,x), x\in[x_0,x_1],$$
$$z_2(t,x_0)=0, t\in[t_1,t_2]. \tag{11}$$

Assuming that $\psi_i(t,x), i=1,2$ are as yet unknown n-dimensional vector functions, we introduce the notation

$$\Delta_{\bar u_i}f_i[t,x]\equiv f_i\big(t,x,\bar u_i(t,x)\big)-f_i\big(t,x,u_i(t,x)\big),$$
$$H(t,x,u_i,\psi_i)=\psi_i' f_i(t,x,u_i),$$
$$\Delta_{\bar u_i}H[t,x,\psi_i]\equiv H(t,x,\bar u_i,\psi_i)-H(t,x,u_i,\psi_i).$$

From relations (8), (10) we obtain that

$$\int_{t_{i-1}}^{t_i}\int_{x_0}^{x_1}\psi_i'(t,x)\frac{\partial^2\Delta z_i(t,x)}{\partial t\partial x}dtdx=$$
$$=\int_{t_{i-1}}^{t_i}\int_{x_0}^{x_1}\Bigg[\psi_i'(t,x)\Bigg(A_i(t,x)\Delta z_i(t,x)+B_i(t,x)\frac{\partial\Delta z_i(t,x)}{\partial t}+$$
$$+C_i(t,x)\frac{\partial\Delta z_i(t,x)}{\partial x}\Bigg)+\Delta_{\bar u_i}H[t,x,\psi_i]\Bigg]dtdx, i=\overline{1,2}. \tag{12}$$

Taking into account boundary conditions (9), (11), we obtain that

$$\Delta z_1(t,x)=\int_{t_0}^{t}\int_{x_0}^{x}\frac{\partial^2\Delta z_1(\tau,s)}{\partial\tau\partial s}d\tau ds, \tag{13}$$
$$\frac{\partial\Delta z_1(t,x)}{\partial t}=\int_{x_0}^{x}\frac{\partial^2\Delta z_1(t,s)}{\partial t\partial s}ds, \tag{14}$$
$$\frac{\partial\Delta z_1(t,x)}{\partial x}=\int_{t_0}^{t}\frac{\partial^2\Delta z_1(\tau,x)}{\partial\tau\partial x}d\tau, \tag{15}$$
$$\Delta z_2(t,x)=B(x)\Delta z_1(t_1,x)+\int_{t_1}^{t}\int_{x_0}^{x}\frac{\partial^2\Delta z_2(\tau,s)}{\partial\tau\partial s}d\tau ds, \tag{16}$$
$$\frac{\partial\Delta z_2(t,x)}{\partial t}=\int_{x_0}^{x}\frac{\partial^2\Delta z_2(t,s)}{\partial t\partial s}ds, \tag{17}$$
$$\frac{\partial\Delta z_2(t,x)}{\partial x}=\frac{\partial B(x)}{\partial x}\Delta z_1(t_1,x)+B(x)\frac{\partial\Delta z_1(t_1,x)}{\partial x}+\int_{t_1}^{t}\frac{\partial^2\Delta z_2(\tau,x)}{\partial\tau\partial x}d\tau, \tag{18}$$

Further, suppose that $\alpha_i(t,x), i=\overline{1,k}$ are characteristic functions of the rectangles $[t_0,T_i]\times[x_0,X_i]$, $\beta_i(t,x)$; $i=\overline{1,k}$ are characteristic functions of the rectangles $[t_1,\theta_i]\times[x_0,X_i]$, and $\gamma_i(x), i=\overline{1,k}$ are characteristic functions of the segments $[x_0,X_i], i=\overline{1,k}.$

Taking into account identities (13) and (16), we obtain that

$$\Delta z_1(T_i,X_i)=\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\alpha_i(t,x)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt, \tag{19}$$

$$\Delta z_2(\theta_i,X_i)=B(X_i)\Delta z_1(t_1,X_i)+\int\limits_{t_1}^{\theta_i}\int\limits_{x_0}^{X_i}\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt=$$
$$=\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{x_1}B(X_i)\gamma_i(x)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt+\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{x_1}\beta_i(t,x)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt. \tag{20}$$

Given identities (13)-(20), the formula for increment (7) of objective functional (6) can be represented as

$$\Delta S(u_1,u_2)=\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\sum_{i=1}^{k}c_i{}'\,\alpha_i(t,x)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt+$$
$$+\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\sum_{i=1}^{k}d_i{}'B(X_i)\gamma_i(x)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt+\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{x_1}\sum_{i=1}^{k}\beta_i(t,x)d_i{}'\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt+$$
$$+\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\psi_1'(t,x)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt-\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\left[\int\limits_{t}^{t_1}\int\limits_{x}^{x_1}\psi_1'(\tau,s)A_1(\tau,s)dsd\tau\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt-$$
$$-\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\left[\int\limits_{x}^{x_1}\psi_1'(t,s)B_1(t,s)ds\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt-$$
$$-\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\left[\int\limits_{t}^{t_1}\psi_1'(\tau,x)C_1(\tau,x)d\tau\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt+\int\limits_{t_2}^{t_2}\int\limits_{x_0}^{x_1}\psi_2'(t,x)\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt-$$
$$-\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{x_1}\left[\int\limits_{t}^{t_2}\int\limits_{t_1}^{t_2}\psi_2'(\tau,s)C_2(\tau,s)B(s)dsd\tau\right]\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt-$$
$$-\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{x_1}\left[\int\limits_{x}^{x_1}\psi_2'(t,s)B_2(t,s)ds\right]\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt-$$
$$-\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{x_1}\left[\int\limits_{t}^{t_2}\psi_2'(\tau,x)C_2(\tau,x)d\tau\right]\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt-$$
$$-\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\Delta_{\bar{u}_1(t,x)}H[t,x,\psi_1]dxdt-\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{x_1}\Delta_{\bar{u}_2(t,x)}H[t,x,\psi_2]dxdt. \tag{21}$$

If we assume that $\psi_i(t,x), i=1,2$ satisfy the relations

$$\psi_1(t,x) = -\sum_{i=1}^{k} c_i d_i(t,x) - \sum_{i=1}^{k} \gamma_i(x) B'(X_i) d_i + \int_t^{t_1}\int_x^{x_1} \psi_1'(\tau,s) A_1(\tau,s) ds d\tau +$$
$$+\int_x^{x_1} \psi_1'(t,s) B_1(t,s) ds + \int_t^{t_1} \psi_1'(\tau,x) C_1(\tau,x) d\tau + \int_{t_1}^{t_2}\int_x^{x_1} B'(s) A_2(\tau,s) ds d\tau +$$
$$+\int_{t_1}^{t_2} B'(x) C_2(\tau,x)\psi_2(\tau,x) d\tau + \int_{t_1}^{t_2}\int_x^{x_1} \frac{B'(s)}{\partial s} C_2(\tau,s)\psi_2(\tau,s) ds d\tau, \tag{22}$$
$$\psi_2(t,x) = -\sum_{i=1}^{k} \beta_i(t,x) d_i + \int_t^{t_2}\int_x^{x_1} A_2(\tau,s)\psi_1(\tau,s) ds d\tau + \int_x^{x_1} \psi_2'(t,s) B_2(t,s) ds$$
$$+\int_t^{t_2} \psi_2'(\tau,x) C_2(\tau,x) d\tau. \tag{23}$$

then the formula for increment (21) takes the form

$$\Delta S(u_1,u_2) = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\overline{u}_1(t,x)} H[t,x,\psi_1] dx dt - \int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{\overline{u}_2(t,x)} H[t,x,\psi_2] dx dt. \tag{24}$$

As we can see, relations (22), (23) represent a system of linear Volterra integral equations with respect to $\psi_1(t,x), \psi_2(t,x)$, respectively. Using formula (24), we prove

Theorem 1. Under the assumptions made, for the optimality of the admissible control $\big(u_1(t,x), u_2(t,x)\big)$ it is necessary and sufficient that the inequalities

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\overline{u}_1(t,x)} H[t,x,\psi_1] dx dt \le 0, \tag{25}$$
$$\int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{\overline{u}_2(t,x)} H[t,x,\psi_2] dx dt \le 0 \tag{26}$$

hold for all $v_i(t,x) \in U_i, (t,x) \in D_i, i = 1,2.$

Proof. Necessity. Suppose that $\big(u_1(t,x), u_2(t,x)\big)$ is an optimal control. Then we have that

$$S\big(v_1(t,x), u_2(t,x)\big) - S\big(u_1(t,x), u_2(t,x)\big) =$$
$$= -\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{v_1(t,x)} H[t,x,\psi_1] dx dt \ge 0,$$
$$S\big(u_1(t,x), v_2(t,x)\big) - S\big(u_1(t,x), u_2(t,x)\big) =$$
$$= -\int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{v_2(t,x)} H[t,x,\psi_1] dx dt \ge 0.$$

The last inequalities imply the correctness of relations (25), (26).

Sufficiency. Suppose that relations (25), (26) hold. Then it follows from increment formula (24) that

$$S(v_1,v_2) - S(u_1,u_2) =$$

$$= -\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{v_1(t,x)} H[t,x,\psi_1]dxdt - \int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{v_2(t,x)} H[t,x,\psi_1]dxdt \geq 0. \quad (27)$$

This proves the sufficiency, since by virtue of (25), (26)

$$S(u_1,u_2) \leq S(v_1,v_2)$$

for arbitrary $\left(v_1(t,x), v_2(t,x)\right)$.

4. The case of a nonlinear convex functional

Suppose that it is required to find the minimum value of the functional

$$S(u_1,u_2) = \varphi_1\left(z_1(T_1,X_1),\dots,z_1(T_k,X_k)\right) + \varphi_2\left(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\right) \quad (28)$$

under constraints (1)-(5).

Here, $\varphi_1(a_1,\dots,a_k), \varphi_2(b_1,\dots,b_k)$ are specified continuously differentiable, convex functions.

Using the Taylor formula, we write the increment of the quality functional corresponding to two admissible controls.

$$\Delta S(u_1,u_2) = \sum_{i=1}^{k} \frac{\partial \varphi_i{}'\left(z_1(T_1,X_1),\dots,z_1(T_k,X_k)\right)}{\partial z_1} \Delta z_1(T_i,X_i) +$$

$$+ \sum_{i=1}^{k} \frac{\partial \varphi_2{}'\left(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\right)}{\partial z_2} \Delta z_2(\theta_i,X_i) +$$

$$+ o_1\left(\sum_{i=1}^{k} \|\Delta z_1(T_i,X_i)\|\right) + o_2\left(\sum_{i=1}^{k} \|\Delta z_2(\theta_i,X_i)\|\right). \quad (29)$$

Using identities (19), (20), we prove that

$$\sum_{i=1}^{k} \frac{\partial \varphi_1{}'\left(z_1(T_1,X_1),\dots,z_1(T_k,X_k)\right)}{\partial z_1} \Delta z_1(T_i,X_i) =$$

$$= \int_{t_0}^{t_1}\int_{x_0}^{x_1} \sum_{i=1}^{k} \alpha_i(t,x) \frac{\partial \varphi_1{}'\left(z_1(T_1,X_1),\dots,z_1(T_k,X_k)\right)}{\partial z_1} \frac{\partial^2 \Delta z_1(t,x)}{\partial t \partial x} dxdt +$$

$$+ o_1\left(\sum_{i=1}^{k} \|\Delta z_1(T_i,X_i)\|\right), \quad (30)$$

$$\sum_{i=1}^{k} \frac{\partial \varphi_2{}'\left(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\right)}{\partial z_1} \Delta z_2(\theta_i,X_i) =$$

$$= \int_{t_1}^{t_2}\int_{x_0}^{x_1} \sum_{i=1}^{k} \beta_i(t,x) \frac{\partial \varphi_2{}'\left(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\right)}{\partial z_1} \cdot \frac{\partial^2 \Delta z_2(t,x)}{\partial t \partial x} dxdt +$$

$$+ \int_{t_1}^{t_2}\int_{x_0}^{x_1} \sum_{i=1}^{k} \gamma_i(x) \frac{\partial \varphi_2{}'\left(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\right)}{\partial z_2} B(X_i) \frac{\partial^2 \Delta z_1(t,x)}{\partial t \partial x} dxdt +$$

$$+o_1\left(\sum_{i=1}^{k}\|\Delta z_2(\theta_i, X_i)\|\right). \tag{31}$$

Suppose $p_i(t,x), i=\overline{1,2}$ are as yet unknown n-dimensional vector functions. Introducing the notation

$$\Delta_{\bar{u}_i}M[t,x,\psi_i] \equiv M(t,x,\bar{u}_i,p_i) - M(t,x,u_i,p_i), i = 1,2$$

and taking into account identities (13)-(18), (30), (31), the formula for increment (29) of functional (28) is written in the form

$$\Delta S(u_1,u_2) = \int_{t_0}^{t_1}\int_{x_0}^{x_1}\sum_{i=1}^{k}\alpha_i(t,x)\frac{\partial \varphi_1{}'\big(z_1(T_1,X_1),\dots,z_1(T_k,X_k)\big)}{\partial z_1}\cdot\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt +$$
$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\sum_{i=1}^{k}\gamma_i(x)\frac{\partial \varphi_2{}'\big(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\big)}{\partial z_2}\cdot B(X_i)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt +$$
$$+\int_{t_1}^{t_2}\int_{x_0}^{x_1}\sum_{i=1}^{k}\beta_i(t,x)\frac{\partial \varphi_2{}'\big(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\big)}{\partial z_1}\cdot\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt +$$
$$+o_1\left(\sum_{i=1}^{k}\|\Delta z_1(T_i,X_i)\|\right)+o_2\left(\sum_{i=1}^{k}\|\Delta z_2(\theta_i,X_i)\|\right)+\int_{t_0}^{t_1}\int_{x_0}^{x_1}p_1{}'(t,x)\cdot\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{t}^{t_1}\int_{x}^{x_1}p_1'(\tau,s)A_1(\tau,s)dsd\tau\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{x}^{x_1}p_1'(t,s)B_1(t,s)ds\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{t}^{t_1}p_1'(\tau,x)C_1(\tau,x)d\tau\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\Delta_{\bar{u}_1(t,x)}M[t,x,p_1]dxdt + \int_{t_2}^{t_2}\int_{x_0}^{x_1}p_2'(t,x)\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\left[\int_{t}^{t_2}\int_{x}^{x_1}p_2'(\tau,s)A_2(\tau,s)B(s)dsd\tau\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{t_1}^{t_2}\int_{x}^{x_1}p_2'(\tau,s)A_2(\tau,s)B(s)dsd\tau\right]\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\left[\int_{x}^{x_1}p_2'(t,s)B_2(t,s)ds\right]\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{t_1}^{t_2}\int_{x}^{x_1}p_2'(\tau,s)C_2(\tau,s)\frac{\partial B(s)}{\partial s}dsd\tau\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{t_1}^{t_2} p_2'(\tau,x)C_2(\tau,x)B(x)d\tau\right]\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\left[\int_{t}^{t_2} p_2'(\tau,x)C_2(\tau,x)d\tau\right]\frac{\partial^2\Delta z_2(t,x)}{\partial t\partial x}dxdt-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\Delta_{\overline{u}_2}M[t,x,p_2]dxdt. \quad (32)$$

Assume that $p_i(t,x), i=\overline{1,k}$ are the solution to the linear Volterra integral equations

$$p_1(t,x)=-\sum_{i=1}^{k}\alpha_i(t,x)\frac{\partial\varphi_1{}'\big(z_1(T_1,X_1),\dots,z_1(T_k,X_k)\big)}{\partial z_1}-$$

$$-\sum_{i=1}^{k}\gamma_i(x)\,B(X_i)\frac{\partial\varphi_2{}'\big(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\big)}{\partial z_2}+\int_{t}^{t_1}\int_{x}^{x_1}p_1'(\tau,s)A_1(\tau,s)dsd\tau+$$

$$+\int_{x}^{x_1}p_1'(t,s)B_1(t,s)ds+\int_{t}^{t_1}p_1'(\tau,x)C_1(\tau,x)d\tau+\int_{t_1}^{t_2}\int_{x}^{x_1}p_2'(\tau,s)A_2(\tau,s)B(s)dsd\tau+$$

$$+\int_{t_1}^{t_2}p_2'(\tau,x)C_2(\tau,x)B(x)d\tau+\int_{t_1}^{t_2}\int_{x}^{x_1}p_2'(\tau,s)C_2(\tau,s)\frac{\partial B(s)}{\partial s}dsd\tau. \quad (33)$$

$$p_2(t,x)=-\sum_{i=1}^{k}\beta_i(t,x)\frac{\partial\varphi_2{}'\big(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\big)}{\partial z_1}+\int_{t}^{t_2}\int_{x}^{x_1}p_2'(\tau,s)A_2(\tau,s)B(s)dsd\tau$$

$$+\int_{x}^{x_1}p_2'(t,s)B_2(t,s)ds+\int_{t}^{t_2}p_2'(\tau,x)C_2(\tau,x)d\tau. \quad (34)$$

Then the formula for increment (32) takes the form

$$\Delta S(u_1,u_2)=-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\Delta_{\overline{u}_1}M[t,x,p_1]dxdt-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\Delta_{\overline{u}_2}M[t,x,p_2]dxdt+$$

$$+o_1\left(\sum_{i=1}^{k}\|\Delta z_1(T_i,X_i)\|\right)+o_2\left(\sum_{i=1}^{k}\|\Delta z_2(\theta_i,X_i)\|\right). \quad (33)$$

By virtue of the convexity of the functions $\varphi_i(.), i=1,2$, it follows from (33) that

$$\Delta S(u_1,u_2)\geq-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\Delta_{\overline{u}_1}M[t,x,p_1]dxdt-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\Delta_{\overline{u}_2}M[t,x,p_2]dxdt. \quad (34)$$

Using inequality (34), we prove

Theorem 2. For the optimality of the admissible control $\big(u_1(t,x),u_2(t,x)\big)$ in problem (1)-(5), (28) it is necessary and sufficient that the inequalities

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1}\Delta_{\overline{u}_1(t,x)}M[t,x,p_1]dxdt\leq 0,$$

$$\int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{\overline{u}_2(t,x)} M[t,x,p_2]dxdt \leq 0,$$

hold for all $v_i(t,x) \in U_i, (t,x) \in D_i, i = 1,2$, respectively.

5. Conclusion

Using a version of the method of increments, a formula for the increment of the multi-point quality functional has been constructed. The constructed increment formula has made it possible to prove a necessary and sufficient optimality condition of the Pontryagin maximum principle type in the problem under consideration.

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