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Mathematical modeling of the deformation of a cylindrical ring under the influence of an external load distributed along the generatrix

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ABSTRACT

The article is devoted to the mathematical modeling of the deformation of a cylindrical ring under the influence of an external load applied to it by pressing parallel plates. It is shown that at the last stage of oil field operation, the main method of artificial lift is the use of sucker rod pumping units (SRPU), on whose failure-free operation the profitability of oil production largely depends. It is also shown that the technical condition of the SRPU is controlled by the dynamometry method, which is the dependence of the rod string force on the suspension on the stroke of the polished rod. In this case, the main source of information is the force transducer. In the article, we propose using a hollow thick-walled cylinder, which is compressed between traverses consisting of parallel flat plates, as the primary sensitive element of the force transducer. The question of refinement and determination of the non-Hertzian contact deformation during compression of a thick-walled hollow cylinder by two parallel flat plates within the elasticity range is considered. It is shown that the well-known formula for the convergence of the axes of the cylinders being compressed according to the Hertzian method does not take into account the real deformation and convergence of the points of the cylinder due to compression and contact deformation between the generatrices of the cylinders. By solving the differential equation, formulas are derived to determine the value of the elastic deformation of a cylinder depending on its geometric dimensions under the influence of an external load applied to it by pressing plates. An example of calculating elastic deformation for different values of cylinder thicknesses depending on the load is given and its graph is plotted.

1. Introduction

According to the Energy Strategy of the Russian Federation until 2030 [1], one of the main problems in the development of the oil and gas complex is inefficient natural resources management, since basically all fields are depleted, and oil production has moved to its last stage of operation – artificial lift. To achieve the strategic goals of the industry development, it is necessary

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to develop and implement modern methods of enhanced oil recovery. It should be noted that this problem is relevant not only for Russia (oil recovery factor in 2008 was 30%), but also for the main oil-producing countries, where the average rated oil recovery is also about 30% [2,3].

At the last stage of oilfield operation, the main method of artificial lift is the use of sucker rod pumping units (SRPU) and electrical submersible pumps (ESP). Thus, in the Russian Federation, 41% of wells are operated by SRPU and 54% by ESP [4]. According to published data, more than 85% of artificial lift wells in the United States are equipped with SRPUs [5]. SRPU is one of the most versatile artificial lift methods.

The main elements of the SRPU automation systems are dynamometer sensors, since it is they that allow obtaining a dynamometer chart containing the necessary information for analyzing malfunctions, both of the submersible pump and the operating mode of the unit as a whole.

Dynamometer sensors are known to be stroke and force transducers. Stroke transducers are designed to determine the motion parameters: suspension points of the sucker rod string – stroke length, swing time and moments when the rod passes the bottom and top dead points. And the force transducers gauge the load on the polished rod, which consists of the weight of the fluid lifted from the well, the weight of the sucker rod string, and the frictional forces in the pump and the string.

The development of SRPU dynamometer sensors is a very important, complex and complex task that requires the use of the most modern approaches in such areas of technology as tensiometry, microelectronics, mathematical signal processing, metrology and reliability. Such sensors must operate continuously over long periods of time within a wide temperature range, withstand the effects of moisture, sulfur and other destructive factors, be resistant to overloads and guarantee operation in hazardous areas [6].

In [7], the results of the development of an inter-traverse intelligent force transducer for SRPU are presented, where it is proposed to measure the value of the deformation of a cylindrical steel ring horizontally under the action of the vertical force of the rod string. Important requirements here are:

– the absence of permanent deformation of the sensing element when the load is removed, which is very important for the SRPU, since on this unit, during the swing time of the head (4-20) per minute, the load on the sensor is applied and removed;

– a deformable sensing element with allowable stresses must provide sufficient deformation and elasticity to achieve high accuracy of the sensor.

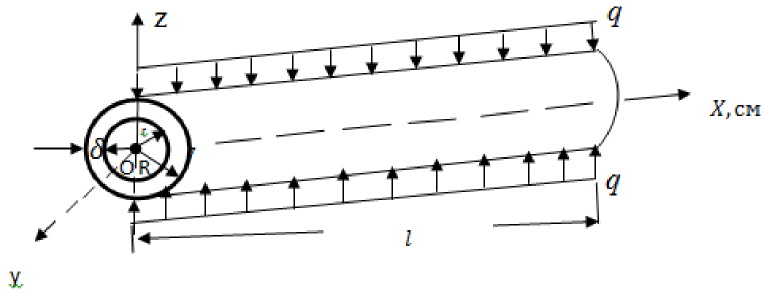
To fulfill these requirements, it is necessary to develop a method for calculating the deformation value of the cylindrical ring depending on its geometric dimensions, under the influence of an external load distributed along the generatrix.

To identify the differences in the terms contact deformation and compression of cylinders, recall that contact deformation is the deformation of bodies in the zone of their contact, in this case, cylinders. Contact deformation is associated with a change in the shape of a cylinder at the point of contact with another body. Due to contact deformation, a local transformation of the round shape of the cylinder into a flat one occurs and a contact area is formed.

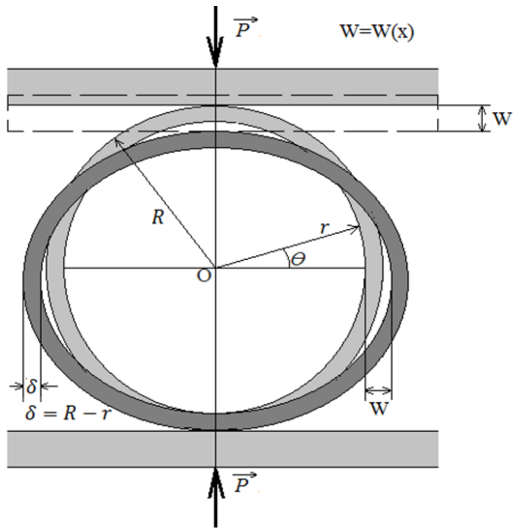
It is known that in H. Hertz's contact problem [1, 2, 3, 10, 11, 12, 13], the stress-strain state in the contact zone of two bodies bounded by surfaces of ellipsoidal type was considered. The considered deformable volume of the bodies was limited to a small zone covering the place of the initial contact of the surfaces of the bodies. The finite dimensions of the bodies themselves were not taken into account, and the bodies were considered as semi-infinite spaces (semi-infinite plane). By virtue of the assumptions made about the smallness of the contact area, the fixed points of the bodies were at an infinitely remote distance.

2. Problem statement

We consider the problem of determining the value of the elastic deformation of a hollow cylindrical ring under compression along the generatrix between two parallel rigid plates. The action of parallel plates is modeled as external loading of a cylinder along a uniformly distributed generatrix.



a)



b)

Fig. 1. Design diagram of the deformation of the elastic element. a) shows the distributed load *q* along the length *l* in coordinates, b) shows the deflection w (displacement) of the middle surface of the shell in elastic mode, $r = const$ the inner radius, see $R = const$ the outer radius, see $\delta = R - r = const$, the shell thickness is constant, along the entire length

The shell material is assumed to be homogeneous and isotropic. The body of the shell is elastically deformed. The load evenly distributed along the generatrix of the cylinder per unit length. When considering a specific section, it can be replaced with $P = q \cdot l -$ a point load.

In this problem, a cylindrical shell is compressed between two rigid parallel plates. The length of the cylindrical shell is taken as q, the ends of which are free from any mechanical loads and moments. It is required to determine the value of elastic deformation along the central, horizontal axis of the cylindrical ring, depending on its geometric dimensions under the influence of an external load applied vertically to it (see Fig. 1b).

3. Solution

We consider the membrane theory of shells, where the radial throw $w(x)$ is described by the equation:

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = \frac{q}{D} \tag{1}$$

where $\beta^4 = \frac{E\delta}{4r^2 D}$ and $D = \frac{E\delta^3}{12(1-\nu^2)}$ is the flexural rigidity of the shell.

Then we have:

$$\beta = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{r\delta}}; \tag{2}$$

where, ν is the Poisson's ratio, and E is Young's modulus of the shell material. Since the Poisson's ratio for steel is taken ν=0.3. Then, for a steel ring, the value of β, depending on the geometric dimensions of the ring, will take the form:

$$\beta = \frac{1{,}285}{\sqrt{r\delta}}, \tag{3}$$

A partial solution to equation (1) represents the deflection (displacement) of the middle surface of the shell, free from fastening, under the action of a uniformly distributed external load along the length *l*. Conditions at the edges of the shell *x = 0* and *x = 1*.

$$w|_{\substack{x=0\\x=l}} = 0, \quad w'|_{\substack{x=0\\x=l}} = 0 \tag{4}$$

We find a solution to equation (1) satisfying boundary conditions (4) in the form:

$$w(x) = \frac{r^2(2-\nu)\cdot q}{2E\delta}\left(1 - e^{-\beta x}(cos\beta x + sin\beta x)\right) \tag{5}$$

The bending moment M_x (per unit length) is found from the relation

$$M_x = D \cdot w''(x) = \frac{q(2-\nu)}{4\beta^2} \cdot e^{-\beta x}(cos\beta x + sin\beta x) \tag{6}$$

The resulting axial force N_x is constant along the length

$$N_x = 0{,}5 \cdot q \cdot \delta \cdot r \tag{7}$$

The resulting circumferential force is determined from the formula

$$N_\theta(x) = qr(1 - (1 - 0{,}5 \cdot \nu))e^{-\beta x}(cos\beta x + sin\beta x) \tag{8}$$

For long shells with $\beta x > 3,$ we have that $e^{-\beta x} < 0{,}05$ and can consider that

$$w(x) = w_*(x) = \frac{q(2-\nu)}{2E\delta} \tag{9}$$

$N_0(x) = qr$. and we can assume that the axial deformation is not constant.

In these expressions, the effect of the affixing of the shells does not show. The calculated relations are also valid for the edge $x = l$, if the distance is counted starting from this edge, i.e., by replacing x with $x_1 = l - x$. The shell strength model contains a separate estimate of general and local stresses (which occurs in the zone of the edge effect).

In the meridional section, the stresses and moments are calculated from the formulas [2, 3]:

$$\sigma_\theta = \sigma_{\theta 0} + \sigma_{\theta z} = \frac{N_\theta}{\delta} - \frac{12 M_\theta z}{\delta^3}, \tag{10}$$

$M_\theta = \nu \cdot M_x; \quad r < z < R;$ where *Z* is the radial coordinate.

To accomplish this task, the cylindrical shell must be deformed elastically, i.e., $\sigma_\theta < [\sigma_{\theta доп}], \quad \sigma_x < [\sigma_{x доп}]$.

For different materials, the admissible values of axial and circumferential stresses at constant loads are given in all relevant manuals of the machine builder. Note that with repeated (cyclic) loads, the role of local stresses increases.

The solution for short beams turns out to be much more complicated, as it is required to take into account the conditions at both ends of the beam. According to [9], it is required to start from a general integral containing four arbitrary constants. These functions, called Krylov functions, are solutions of the homogeneous equation (1) and satisfy special conditions for.

Let us compile the following table, which summarizes the initial values of the Krylov functions and their derivatives:

$$\left.\begin{array}{ccccc} K_k(0) & K'_k(0) & K''_k(0) & K'''_k(0) & \\ K_1 & 1 & 0 & 0 & 0 \\ K_2 & 0 & 1 & 0 & 0 \\ K_3 & 0 & 0 & 1 & 0 \\ K_4 & 0 & 0 & 0 & 1 \end{array}\right\}$$

Since in all cells of this table there are zeros only on the main diagonal of the unit, the system of partial solutions K_k is called a system with a unit matrix. The essence of these solutions is as follows:

$$\left.\begin{aligned} K_1(\beta x) &= \operatorname{ch}\beta x\cdot\cos\beta x\,; \\ K_2(\beta x) &= \frac{1}{2\beta}[\operatorname{ch}\beta x\cdot\sin\beta x+\operatorname{sh}\beta x\cdot\cos\beta x]; \\ K_3(\beta x) &= \frac{1}{2\beta^2}\operatorname{sh}\beta x\cdot\sin\beta x\,; \\ K_4(\beta x) &= \frac{1}{4\beta}[\operatorname{ch}\beta x\cdot\sin\beta x-\operatorname{sh}\beta x\cdot\cos\beta x]. \end{aligned}\right\}$$

It should be noted that the derivatives of the Krylov functions are expressed again in terms of the same functions, and:

$$\left.\begin{aligned} K'_1 &= -4K_4; \\ K'_2 &= K_1; \\ K'_3 &= K_2; \\ K'_4 &= K_3; \end{aligned}\right\}$$

Thus, the general integral of equation (1) can be represented through the Krylov functions:

$$w(x) = C_1K_1(\beta x)+C_2K_2(\beta x)+C_3K_3(\beta x)+C_4K_4(\beta x)+K_*(x).$$

Constant integrations of C_1, C_2, C_3, C_4 have a very definite meaning here. Indeed, if we assume x =0 and use the property of the introduced functions, we get:

$$\left.\begin{aligned} C_1 &= w(0)\,; \\ C_2 &= w'(0)\,; \\ C_3 &= w''(0)\,; \\ C_4 &= w'''(0) \end{aligned}\right\}$$

Thus:

$$w(\beta x) = w(0)K_1(\beta x)+w'(0)K_2(\beta x)+w''(0)K_3(\beta x)+$$

$$w'''(0)K_4(\beta x)+K_*(\beta x).$$

This formula represents the general integral of equation (1). Integration constants have a simple meaning here: these are the initial (for $x = 0$) values of the required function and its derivatives. Therefore, the method of integrating differential equations based on the formula is widely used in structural mechanics and is called the method of initial parameters. Considering that $\beta = const$, - we replace βx with x.

Then, according to [9], the solution of equation (1) can be written in the final form:

$$w(x) = w(0) \cdot K_1(x) + w'(0) \cdot K_2(x) + w''(0)K_3(x) + w'''(0)K_4(x) + \\ + K_*(x); \quad (11)$$

where the Krylov functions *K(x)* [4,5,9] are expressed by the relations

$$K_1(x) = ch\beta x \cdot cos\beta x;$$

$$K_2(x) = \frac{ch\beta x}{2\beta}(cos\beta x + sin\beta x) \quad (12)$$

$$K_3(x) = (sh\beta x \cdot sin\beta x)/(2\beta^2)$$

$$K_4(x) = (ch\beta x \cdot sin\beta x - sh\beta x \cdot cos\beta x)/(4\beta^3)$$

The function $K_*(x)$ is expressed by the formula

$$K_*(x) = \frac{q(1-v)}{D}\int_0^x K_4(x-\xi) \cdot d\xi \text{ or} \quad (13)$$

$$K_*(x) = \frac{q(1-v)}{8D\beta^2} \cdot (\frac{e^{\beta x}}{1+\beta^2}(\beta sin\beta x - cos\beta x) - \frac{1}{1+\beta^2} - \\ - \frac{2\beta \cdot cos\beta x \cdot sh\beta x}{\beta^2+1} + \frac{2}{1+\beta^2} \cdot ch\beta x \cdot sin\beta x) \quad (14)$$

and is a partial solution of differential equation (1). Here we take into account that in this case the boundary disturbance zone occupies the entire length of the shell. Therefore, the stiffness model for short shells is based on the condition $w(x) < f_0$, where f_0 is admissible deflection value (elastic displacement of the generatrix of the shell).

Given that $q = \frac{P}{l}$ and denoting

$$w_0(x) = w(0) \cdot K_1(x) + w'(0) \cdot K_2(x) + w''(0)K_3(x) + w'''(0)K_4(x); \quad (15)$$

$$w_1(x) = \frac{(1-v)}{8D\beta^2 l\,(1+\beta^2)} \cdot (e^{\beta x}(\beta sin\beta x - cos\beta x) - \\ -2\beta \cdot cos\beta x \cdot sh\beta x + 2 \cdot ch\beta x \cdot sin\beta x) - 1) \quad (16)$$

We obtain that

$$K_*(x) = P \cdot w_1(x);$$

$$w(x) = P \cdot w_1(x) + w_0(x). \quad (17)$$

Whence it follows that the deformation of the axial section of the shell linearly depends on the force of gravity acting on the system.

An example of calculating the elastic deformation of a cylindrical ring with data:

r =5 cm, *R*= *r* + δ, l = 4 cm, ν = 0,3, *E*= 2,1$\cdot$ 10^6kg/cm^2

We calculate the values of the radial deformation of the cylindrical ring for the values of δ ={1,0; 2,0} in the following sequence:

Finding the values $\beta_{\delta=1.0}$, $\beta_{\delta=2.0}$, $D_{\delta=1.0}$, $D_{\delta=2.0}$, $w_{0\delta=1.0}$, $w_{0\delta=2.0}$, $w_{1\delta=1.0}$, $w_{1\delta=2.0}$ taking into account formulas (3), (12), (15), (16) and enter the elastic deformation data into the table. Using the values entered in the table, we build a graph of the change in deformation from the force of gravity acting on the system (Fig. 2).

Table of elastic deformation data

δ. *cm*	1	2
β	0.575	0.41
D	$1.923 \cdot 10^5$	$15.3846 \cdot 10^5$
W_0 , *cm*	0.0032	0.0014
W_1, *cm*	0.002865	0.00048
$W = W_0 + PW_1$, *cm*, *for* $P=2\times10^3 kg$ *for* $P=10\times10^3 kg$	 0.00812 0.03185	 0.0024 0.0062

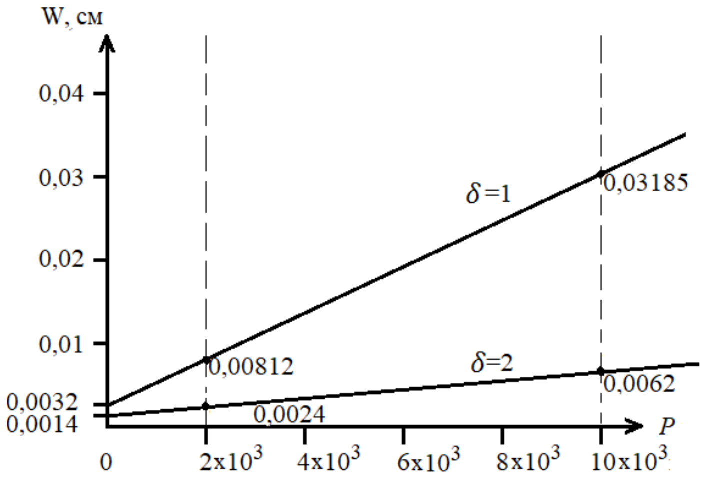


Fig. 2. Graph of deformation versus gravity acting on the system

4. Conclusion

1. As a result of the analytical study, the behavior of a hollow cylindrical ring under compression between two rigid plates under a load uniformly distributed along the generatrix has been described by a differential equation. By solving the differential equation, an analytical expression has been found that describes the dependence of the value of the elastic radial deformation of the cylindrical ring depending on its geometric dimensions and external load applied to it.

2. It has been revealed that the value of the elastic deformation of a cylindrical ring linearly depends on the external load applied to it, which makes it possible to use it as a sensing element of the force transducer gauging the force of the rods on the SRPU suspension in linear displacement.

3. A method has been developed for calculating the value of the elastic deformation of the cylindrical ring depending on its geometric dimensions, under the influence of an external load distributed along the generatrix, which can be used when choosing the parameters of a sensing elastic element.

4. Thus, a mathematical model of the behavior of a cylindrical ring under compression between two rigid plates has been constructed and its structural identification has been carried out.

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