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A model of a queuing system with a base stock policy and two supply sources

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ABSTRACT

We propose a Markov model of a two-source queuing-inventory system with an $(S-1,S)$ inventory control policy. Replenishments from different sources are carried out with different delays. If the inventory level falls to a certain threshold value, an emergency order to the fast source is instantly generated, with the fast source being expensive. Events occur in the system, as a result of which the inventory level instantly decreases. The ergodicity conditions of the system under study are found, its stationary distribution is calculated and formulas for finding and optimizing its characteristics are proposed.

1. Introduction

In classical inventory systems, the assumption is that the time of sale (release) of inventory to consumers is zero [1]. If the service time in these systems is a positive random variable, then the formation of queues of incoming consumer customers (c-customers) is possible. These systems are called Queuing-Inventory Systems (QIS) [2, 3]. The foundations of the QIS theory were laid in [4, 5] and it has been intensively developed over the last three decades by various authors, see the review paper [6].

The present study is a continuation of the one started in [7]. Unlike that paper, here we develop a QIS model that uses the $(S-1,S)$ inventory control policy, in which it is possible to replenish stocks from two sources: a slow one and a fast one, with the fast one being an expensive source. In $(S-1,S)$ policy, every time the inventory levels decreases by one (through sale or spoilage), a unit size inventory replenishment order is placed. It is sometimes called Base Stock Policy or One-To-One Policy. This policy is known to be economically advantageous in those QIS where the inventory is an expensive item and the order size is limited for technical reasons (when the inventory is a bulky item, such as cars, TV or music centers, etc.), and the lead time is quite important (i.e., the order execution rate is low). Here we found the condition of ergodicity of the model and proposed an algorithm for finding its characteristics using the matrix-geometric method [8], and solved the problem of minimizing the total cost of the system by finding the optimal value of the reorder point for the fast and expensive source.

2. Model description

We consider an $M/M/1/\infty$ QIS model with consumer and destructive customers and two inventory replenishment sources using the $(S-1,S)$ ordering policy. The incoming c-customer flow

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is a Poisson flow with rate λ, and for simplicity we assume that these customers require a unit size inventory. Furthermore, the flow of destructive customers (d-customers) is also assumed to be a Poisson flow with parameter κ, with the inventory level instantly decreasing by one when such customers arrive; a d-customer may even destroy the stock that is in the release phase to a c-customer. If the inventory level is zero, the incoming d-customer has no effect on the system.

If the server is idle and the inventory level is positive at the time the c-customer arrives, it is immediately accepted for service; if the inventory level is positive and the server is busy at that time, the received c-customer is put into an infinite-length queue. It is assumed that c-customers can join the queue even when the inventory level is zero; in other words, if the system is out of inventory when the next c-customer arrives, it either becomes queued with probability φ_1, or leaves the system with probability $\varphi_2 = 1 - \varphi_1$. The customer at the top of the queue becomes impatient if the inventory level drops to zero, i.e., in such cases the c-customer at the top of the queue waits for some random time, which has an indicative d.f. with mean τ^{-1}, and after this time it leaves the system with an unsatisfied demand.

After the completion of service, the c-customer either refuses to receive the goods with probability σ_1, or receives the goods with probability $\sigma_2 = 1 - \sigma_1$. In both cases, the service times of c-customers have exponential d.f., but their mean values are different, i.e., if the c-customer refuses to receive the goods, then its average service time is μ_1^{-1}; otherwise, this time is μ_2^{-1}.

The system uses the $(S-1, S)$ inventory control policy, i.e., every time the system's inventory level decreases by one (either as a result of selling the goods to a c-customer or due to their spoilage at the moment of d-customer's arrival) an order is sent to replenish the inventory, with the order size equal to one unit. It is considered that it is possible to replenish the inventory from two sources: slow Source 1 and fast Source 2. An order to the slow source is called a regular order, and an order to the fast source is called an emergency order. The lead time for orders from each source has an exponential d.f., but their mean values are different, i.e., if an order is placed to Source i, then the average waiting time for supply of inventory is $\nu_i^{-1}, i = 1,2$, with $\nu_2 > \nu_1$. This means that supply from Source 2 requires a shorter delivery time than from Source 1, but supply from Source 2 requires additional costs.

A regular order may be cancelled before it has been executed, and the time required to formalize the act of cancellation is zero. An emergency order is issued if the inventory level drops to a value of $r,\ 1 \le r \le S-1$, i.e., when the inventory level drops below this value, the order is instantly sent to Source-2. In this case, it is considered that the cancellation of the order from Source 1 is associated with certain penalties.

The problem is to find the joint distribution of the number of c-customers in the system and the inventory level of the system, to determine the main characteristics of the system, and to solve the problem of choosing the optimal threshold value of r in order to minimize the total cost of the system when using both ordering schemes for inventory replenishment. The total cost includes the variable and fixed costs for each type of order and order cancellation, as well as the costs of inventory storage, penalties due to the loss of c-customers due to their impatience, and penalties for their dwell in the system.

3. Calculation of probabilities of system states

The system is described by a two-dimensional Markov chain (2-DMC) with states of the form (n, m), where n indicates the number of c-customers in the system, $n = 0,1,...$, and m denotes the inventory level in the warehouse of the system, $m = 0,1,..., S$. The state space (SS) of this 2-DMC is determined as follows:

$$E = \bigcup_{n=0}^{\infty} L(n)$$

where the set $L(n) = \{(n, 0), (n, 1),..., (n, S)\}$ is called the n-th level.

Suppose that the initial state of the system is $(n_1, m_1) \in E$. Then possible transitions between states and their rates are determined as follows.

- If a c-customer arrives and the inventory level is zero, then the transition from this state to the state $(n_1 + 1{,}0) \in E$ occurs at the rate $\lambda\varphi_1$.
- If a c-customer arrives and the inventory level is greater than zero, then the transition from this state to the state $(n_1 + 1, m_1) \in E$ occurs at the rate λ.
- If, after the completion of service, the c-customer refuses to receive the goods, then the transition from this state to the state $(n_1 - 1, m_1) \in E$ occurs at the rate $\mu_1\sigma_1$.
- If, after the completion of service, the c-customer receives the goods, then the transition from this state to the state $(n_1 - 1, m_1 - 1) \in E$ occurs at the rate $\mu_2\sigma_2$.
- If in the initial state $(n_1, m_1) \in E$ we have $m_1 = 0$, then the transition from this state to the state $(n_1 - 1{,}0) \in E$ occurs at the rate τ .
- If the inventory level is greater than zero at the arrival of a d-customer, then the transition from this state to the state $(n_1, m_1 - 1) \in E$ occurs at the rate κ.
- If, at the time of replenishment, the inventory level is $m_1, 0 \le m_1 \le r - 1$, then the transition to the state $(n_1, m_1 + 1) \in E$ occurs at the rate $(r - m_1)\nu_2$.
- If, at the time of replenishment, the inventory level is $m_1, r \le m_1 \le S - 1$, then the transition to the state $(n_1, m_1 + 1) \in E$ occurs at the rate $(S - m_1)\nu_1$.

The rate of transition from the state $(n_1, m_1) \in E$ into another state $(n_2, m_2) \in E$ will be denoted by $q((n_1, m_1), (n_2, m_2))$. Then we conclude that the positive elements of the generator of the studied 2-DMC are determined as follows:

$$q\big((n_1,m_1),(n_2,m_2)\big) = \begin{cases} \lambda\varphi_1, & if\ n_2 = n_1 + 1,\ m_2 = m_1 = 0, \\ \lambda, & if\ n_2 = n_1 + 1,\ m_2 = m_1 > 0, \\ \mu_1\sigma_1, & if\ n_2 = n_1 - 1,\ m_2 = m_1 > 0, \\ \mu_2\sigma_2, & if\ n_2 = n_1 - 1, m_2 = m_1 - 1, \\ \kappa, & if\ n_2 = n_1,\ m_1 > 0,\ m_2 = m_1 - 1, \\ \tau, & if\ n_1 > 0,\ n_2 = n_1 - 1,\ m_2 = m_1 = 0, \\ (r - m_1)\nu_2, & if\ n_2 = n_1, 0 \le m_1 \le r - 1,\ m_2 = m_1 + 1, \\ (S - m_1)\nu_1, & if\ n_2 = n_1, r \le m_1 \le S - 1,\ m_2 = m_1 + 1. \end{cases} \quad (1)$$

Based on relations (1), we conclude that the obtained 2-DMC is a Level Independent Quasi-Birth-Death Process (LIQBD). By renumbering the states of this 2-D MC lexicographically (i.e., they are numbered in the order of $(0{,}0), (0{,}1),..., (0, S), (1{,}0), (1{,}1),..., (1, S),...$), we conclude that the generator of the obtained LIQBD looks as follows:

$$G = \begin{pmatrix} B & A_0 & O & O & O & \dots \\ A_2 & A_1 & A_0 & O & O & \dots \\ O & A_2 & A_1 & A_0 & O & \dots \\ O & O & A_2 & A_1 & A_0 & \dots \\ O & O & O & A_2 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (2)$$

In (2), O denotes a zero square matrix of size $S+1$, and $B = \|b_{ij}\|$, $A_k = \left\|a_{ij}^{(k)}\right\|, i,j = 0,1,...,S,$ are square matrices of the same size, their nonzero elements being determined from the following relations:

$$b_{ij} = \begin{cases} (r-i)\nu_2, & if\ 0 \le i \le r-1, j = i+1, \\ (S-i)\nu_1, & if\ r \le i \le S-1,\ j = i+1, \\ \kappa, & if\ i > 0,\ j = i-1, \\ -(r\nu_2 + \lambda\varphi_1), & if\ i = j = 0, \\ -((r-i)\nu_2 + \kappa + \lambda), & if\ 1 \le i \le r-1,\ i = j, \\ -((S-i)\nu_1 + \kappa + \lambda), & if\ r \le i \le S,\ i = j; \end{cases} \quad (3)$$

$$a_{ij}^{(0)} = \begin{cases} \lambda\varphi_1, & if\ i = j = 0, \\ \lambda, & if\ i > 0, i = j; \end{cases} \quad (4)$$

$$a_{ij}^{(1)} = \begin{cases} (r-i)\nu_2, & if\ 0 \le i \le r-1, j = i+1, \\ (S-i)\nu_1, & if\ r \le i \le S-1, j = i+1, \\ \kappa, & if\ i > 0, j = i-1, \\ -(\tau + r\nu_2 + \lambda\varphi_1), & if\ i = j = 0, \\ -((r-i)\nu_2 + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & if\ 1 \le i \le r-1, i = j, \\ -((S-i)\nu_1 + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & if\ r \le i \le S, i = j; \end{cases} \quad (5)$$

$$a_{ij}^{(2)} = \begin{cases} \tau, & if\ i = j = 0, \\ \mu_1\sigma_1, & if\ i > 0, i = j, \\ \mu_2\sigma_2, & if\ i > 0, j = i-1. \end{cases} \quad (6)$$

***Statement* 1.** The system is ergodic if and only if the following relation is satisfied

$$\lambda(1 - \varphi_2\pi(0)) < \tau\pi(0) + (\mu_1\sigma_1 + \mu_2\sigma_2)(1 - \pi(0)), \quad (7)$$

where

$$\pi(0) = \frac{1}{r!}\left(\sum_{m=0}^{r} \frac{\theta_2^m}{(r-m)!} + (S-r)!\left(\frac{\nu_2}{\nu_1}\right)^r \sum_{m=r+1}^{S} \frac{\theta_1^m}{(S-m)!}\right)^{-1},$$
$$\theta_i = \nu_i/(\mu_2\sigma_2 + \kappa), i = 1,2.$$

***Proof*.** We denote the stationary distribution, which corresponds to the generator $A = A_0 + A_1 + A_2$, by $\boldsymbol{\pi} = (\pi(0), \pi(1), ..., \pi(S))$. Note that the quantities $\pi(m),\ m = 0,1,...,S,$ represent the probabilities that the inventory level is $m,\ m = 0,1,...,S$. These quantities are found from the following system of balance equations:

$$\boldsymbol{\pi A} = \mathbf{0},\ \boldsymbol{\pi e} = \mathbf{1}, \quad (8)$$

where **0** is an S+1 zero row vector and ***e*** is an S+1zero row vector, all components of which are equal to one.

From relations (4)-(6) we conclude that the nonzero elements of the generator A are determined as follows:

$$a_{ij}=\begin{cases}-(\tau+r\nu_2+\lambda\varphi_1), & if\ j=i=0,\\ -(\mu_2\sigma_2+\kappa+(r-i)\nu_2), & if\ 1\le i\le r-1, j=i,\\ -(\mu_2\sigma_2+\kappa+(S-i)\nu_1), & if\ r\le i\le S, j=i,\\ \mu_2\sigma_2+\kappa, & if\ 1\le i\le S, j=i-1,\\ (r-i)\nu_2, & if\ 0\le i\le r-1, j=i+1,\\ (S-i)\nu_1, & if\ r\le i\le S-1, j=i+1.\end{cases} \quad (9)$$

From relations (9) we conclude that system of equations (8) is a balance equation for the one-dimensional birth-and-death process, where the death rate is a constant equal to $\mu_2\sigma_2+\kappa$, and the birth rates depend on the system state. Consequently, we have:

$$\pi(m)=\begin{cases}\frac{r!}{(r-m)!}\theta_2^m\pi(0), & if\ 0\le m\le r,\\ r!\,(S-r)!\left(\frac{\nu_2}{\nu_1}\right)^r\frac{\theta_1^m}{(S-m)!}\pi(0), & if\ r+1\le m\le S,\end{cases} \quad (10)$$

where $\pi(0)$ is calculated by means of the normalization condition, i.e., $(\boldsymbol{\pi},\boldsymbol{e}^T)=1$, where $\boldsymbol{e}^T$ is a transposition of the column vector $\boldsymbol{e}$ and $(\boldsymbol{\pi},\boldsymbol{e}^T)$ is the product of the vectors $\boldsymbol{\pi}$ and $\boldsymbol{e}^T$.

The constructed LIQBD is ergodic if and only if the following relation is satisfied (see [8], pp. 81-83):

$$\boldsymbol{\pi}A_0\boldsymbol{e}<\boldsymbol{\pi}A_2\boldsymbol{e}. \quad (11)$$

Hence, taking into account relations (4), (6) and (10), after performing certain transformations, we obtain from (11) that Statement 1 is true.

Then the sought stationary distribution corresponding to the generator G, $\boldsymbol{p}=(\boldsymbol{p}_0,\boldsymbol{p}_1,\boldsymbol{p}_2,\dots)$,$\boldsymbol{p}_n=\left(p(\mathrm{n},0),p(n,1),\dots,p(n,S)\right)$, $n=0,1,\dots$, is determined as

$$\boldsymbol{p}_n=\boldsymbol{p}_0R^n,\ n\ge 1, \quad (12)$$

where R is a non-negative and minimum solution of the following quadratic matrix equation:

$$R^2A_2+RA_1+A_0=0. \quad (13)$$

The probabilities $\boldsymbol{p}_0$ of the boundary states are calculated from the following system of equations:

$$\boldsymbol{p}_0(B+RA_2)=\boldsymbol{0}, \quad (14)$$

$$\boldsymbol{p}_0(I-R)^{-1}\boldsymbol{e}=1, \quad (15)$$

where I is the identity matrix of the size $S+1$.

4. Characteristics of the system

The main characteristics of the system are the following quantities.

- Average inventory level of the system (S_{av})

$$S_{\text{av}}=\sum_{m=1}^{S}m\sum_{n=0}^{\infty}p(n,m) \quad (16)$$

- Average supply volume from Source i, i=1, 2, ($V_{\text{av}}(i)$)

$$V_{\text{av}}(1)=\sum_{n=0}^{\infty}\sum_{m=r}^{S-1}p(n,m);\quad V_{\text{av}}(2)=\sum_{n=0}^{\infty}\sum_{m=0}^{r-1}p(n,m); \quad (17)$$

- Average number of c-customers in the system (L_{av})

$$L_{\text{av}}=\sum_{n=1}^{\infty}n\sum_{m=0}^{S}p(n,m); \quad (18)$$

- Average destruction rate of the stocks (DRS)):

$$\text{DRS} = \kappa(1 - \sum_{n=0}^{\infty} p(n,0)); \tag{19}$$

- Average rate of regular orders (RR_1):

$$\text{RR}_1 = \kappa \sum_{m=r+1}^{S} p(0,m) + (\mu_2\sigma_2 + \kappa) \sum_{n=1}^{\infty} \sum_{m=r+1}^{S} p(n,m)\,; \tag{20}$$

- Average rate of emergency orders (RR_2):

$$\text{RR}_2 = \kappa \sum_{m=1}^{r} p(0,m) + (\mu_2\sigma_2 + \kappa) \sum_{n=1}^{\infty} \sum_{m=1}^{r} p(n,m)\,; \tag{21}$$

Note. The average rate of emergency orders is equal to the average rate of cancellation (RC) of regular orders.

- Probability of loss of c-customers (PL)

$$\text{PL} = \varphi_2 \sum_{n=0}^{\infty} p(n,0) + \frac{\tau}{\tau + \lambda\varphi_1 + r\nu_2} \sum_{n=1}^{\infty} p(n,0)\,. \tag{22}$$

5. Numerical results

The proposed approach allows conducting numerical experiments to study the behavior of characteristics (16)-(22). In this case, any initial (structural or load) parameter of the system can be taken as a controllable parameter. For brevity, the results of calculation of the above characteristics are not given here, but only the task of minimizing the total loss (TC) of the system is considered.

In all experiments, the values of the probabilities φ_k and $\sigma_k,\ k = 1{,}2$, are locked, i.e., it is considered that $\varphi_1 = 0.6$ and $\sigma_1 = 0.4$.

The advantage of the scheme of emergency ordering proposed here is that, unlike the classical $(S-1, S)$ policy with one replenishment source, there is a controllable parameter, *r* (the emergency order point). This makes it possible to influence the economic performance of the system. Proceeding from this fact, the behavior of total loss in the system is studied in relation to the change of values of the specified parameter at fixed values of load parameters.

The total loss is calculated as follows:

$$\text{TC}(r) = \sum_{i=1}^{2}(K_i + c_i)\text{RR}_i\,(r) + c_c\text{RR}_2(r) + c_h S_{\text{av}}(r) + c_d\text{DRS}(r) + c_l\lambda\text{PL}(r) + c_w L_{\text{av}}(r), \tag{23}$$

where c_i is the unit price of the order volume from Source *i*, K_i is the fixed price of a single order from Source *i, i*=1, 2; c_c is the penalties for canceling a single order from Source 1; c_h is the storage price of a unit of inventory volume per unit of time; c_d is the penalties for destroying a unit of stock; c_l is the penalties for loss of one customer; c_w is the price per unit of waiting time in the queue of one *c*-customer.

The optimization problem is to find such a value of the parameter r that minimizes (23). The values of the initial model parameters and coefficients in functional (23) are chosen as follows:

$$\lambda = 20,\ \ \kappa = 10,\ \mu_1 = 35,\ \mu_2 = 25,\ \ \tau = 20,\ \nu_1 = 5,\ \nu_2 = 10,\ \ \sigma_1 = 0.4,\ \phi_1 = 0.6;$$

$$S = 50,\ c_1 = 50,\ c_c = 50,\ c_h = 35,\ c_d = 75,\ c_l = 200,\ c_w = 50,\ K_1 = 100,\ K_2 = 200.$$

Numerical experiments were conducted for different values of c_2 and a portion of their results are shown in the table, where the optimal values of the parameter *r* are highlighted in bold. The table shows that the optimal value of the threshold parameter r increases as the order price from a fast and expensive source increases. This result is consistent with the theoretical expectations.

Table.

The results of solving the problem of minimizing TC.

r	*TC*	
	c_2=60	c_2=100
1	4316,405	4316,408
2	4316,324	**4316,397**
3	4316,517	4316,771
4	4317,027	4317,537
5	4317,667	4318,451
6	4318,297	4319,346
7	4318,894	4320,196
8	**4314,945**	4316,856
9	4315,696	4317,917
10	4316,488	4319,024
…	…	…
48	6538,532	7090,779
49	6664,671	7242,600

6. Conclusion

In the study, we propose a new replenishment policy in queuing-inventory systems, which uses the $(S-1, S)$ inventory control policy and with two sources of supply. In addition to consumer customers, the system has destructive customers, the arrival of which leads to an instantaneous reduction of the system's inventory level. A threshold parameter for the system inventory level is introduced, which determines the moment of cancellation of an order from a slow source and simultaneous generation of a new order to a fast source. The system has an infinite size buffer for waiting for expendable orders, which are impatient if the inventory level drops to zero. A mathematical model of the system using the proposed replenishment policy in the form of a two-dimensional Markov chain is developed and the condition of its ergodicity is found.

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