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# Necessary and sufficient optimality condition in one discrete optimal control problem

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**ABSTRACT**

*The article investigates a linear discrete optimal control problem, which is an analogue of the linear continuous optimal control problem of population dynamics. The necessary and sufficient optimality condition is proved.*

## 1. Introduction

In [1], a problem of optimal control of the population dynamics described by a system of Fredholm integrodifferential equations of the first order was studied. The necessary optimality condition in the form of the variational maximum principle was proved.

In the proposed study, we consider a linear and discrete analogue of the problem from [1].

Applying one variant of the method of increments the necessary and sufficient optimality condition in the form of the discrete maximum principle is proved.

The case of nonlinear convex quality criterion is studied separately.

## 2. Problem statement

Suppose that $U \in R^r$ is a specified non-empty and bounded set, $t_0, t_1, x_0, x_1$ are specified numbers, and the differences $t_1 - t_0,\ x_1 - x_0$ are natural numbers.

We consider the problem of finding the minimum value of the functional

$$S(u) = \sum_{x=x_0}^{x_1} c'(x) z(t_1, x), \tag{1}$$

under constraints

$$z(t+1, x) = A(t,x) z(t,x) + B(t,x) y(t,x) + f\big(t, x, u(t,x)\big),$$

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$$t = t_0, t_0 + 1, \dots, t_1 - 1, x = x_0, x_0 + 1, \dots, x_1, \quad (2)$$

$$z(t_0, x) = a(x), x = x_0, x_0 + 1, \dots, x_1, \quad (3)$$

$$y(t,x) = \sum_{s=x_0}^{x_1} \left[C(t,x,s)z(t,s) + D\big(t,x,s,u(t,s)\big)\right]. \quad (4)$$

Here, $A(t,x), B(t,x), C(t,x,s)$ are specified $(n \times n)$ discrete matrix functions; $a(x)$ is a specified $n$-dimensional discrete vector function; $f(t,x,u), D(x,s,u)$ are specified $n$-dimensional vector functions that are continuous in $u$ and discrete in the rest of the arguments; $\mathrm{c}(x)$ is a specified $n$-dimensional discrete vector function; and $u(t,x)$ is a specified $r$-dimensional discrete vector of control functions satisfying the inclusion constraint

$$u(t,x) \in U \subset R^r, t = t_0, t_0 + 1, \dots, t_1 - 1, x = x_0, x_0 + 1, \dots, x_1. \quad (5)$$

Such control functions will be called admissible.

The admissible control $u(t,x)$ satisfying the minimum value of functional (1) under constraints (2)-(5) will be called an optimal control, and the corresponding process $(u(t,x), z(t,x))$ an optimal process.

As can be seen, equation (2) is a discrete analogue of a linear Fredholm integrodifferential equation, i.e., the process is described by a two-dimensional system of difference equations.

Note that various optimal control problems described by ordinary difference equations have been studied in works [2-4] and others.

### 3. Quality functional increment formula and optimality condition

Suppose that $(u(t,x), z(t,x))$ is some admissible process. We denote an arbitrary admissible process by $(\bar{u}(t,x) = u(t,x) + \Delta u(t,x), \bar{z}(t,x) = z(t,x) + \Delta z(t,x))$ and write the increment of the objective functional. We have

$$\Delta S(u) = S(\bar{u}) - S(u) = \sum_{x=x_0}^{x_1} c'(x) \Delta z(t_1, x). \quad (6)$$

Here, the increment $\Delta z(t,x)$ of the state $z(t,x)$ will be the solution to the problem

$$\Delta z(t+1,x) = A(t,x)\Delta z(t,x) + B(t,x)\Delta y(t,x) + \left[f\big(t,x,\bar{u}(t,x)\big) - f\big(t,x,u(t,x)\big)\right], \quad (7)$$

$$\Delta z(t_0, x) = 0, \quad (8)$$

$$\Delta y(t,x) = \sum_{x=x_0}^{x_1} [C(t,x,s)\Delta z(t,s) + D(t,x,s)\bar{u}(t,s) - D(t,x,s)u(t,s)]. \quad (9)$$

Therefore, from (7) we get that

$$\Delta z(t+1,x) = A(t,x)\Delta z(t,x) + \sum_{s=x_0}^{x_1} B(t,x)C(t,x,s)\Delta z(t,s) +$$

$$+ \sum_{s=x_0}^{x_1} B(t,x)\left[D\big(t,x,s,\bar{u}(t,s)\big) - D\big(t,x,s,u(t,s)\big)\right] +$$

$$+\big[f[t,x,\bar{u}(t,x)] - f[t,x,u(t,x)]\big], \quad (10)$$

$$\Delta z(t_0, x) = 0. \quad (11)$$

We introduce the notation

$$\Delta_{\bar{u}(t,x)}f[t,x] \equiv f(t,x,\bar{u}(t,x)) - f(t,x,u(t,x))$$

$$\Delta_{\bar{u}(t,s)}D[t,x,s] \equiv D\big(t,x,s,\bar{u}(t,s)\big) - D\big(t,x,s,u(t,s)\big). \tag{12}$$

Suppose that $\psi(t,x)$ is as yet unknown $n$-dimensional vector function. From (10), taking into account the introduced notation, we get that

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\psi'(t,x)\Delta z(t+1,x) = \sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\psi'(t,x)A(t,x)\Delta z(t,x) +$$
$$+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\sum_{s=x_0}^{x_1}\psi'(t,x)B(t,x)\mathrm{C}(t,x,s)\Delta z(t,s) +$$
$$+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\sum_{s=x_0}^{x_1}\psi'(t,x)B(t,x)\Delta_{\bar{u}(t,s)}D[t,x,s] +$$
$$+\sum_{t=t_0}^{t_1-1}\sum_{t=t_0}^{x_1}\psi'(t,x)\Delta_{\bar{u}(t,x)}f[t,x]. \tag{13}$$

Hence, by grouping similar terms we obtain that

$$\sum_{t=t_0}^{t_1-1}\sum_{t=t_0}^{x_1}\psi'(t,x)\Delta z(t+1,x) = \sum_{t=t_0}^{t_1-1}\sum_{t=t_0}^{x_1}\psi'(t,x)A(t,x)\Delta z(t,x) +$$
$$+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\sum_{s=x_0}^{x_1}\psi'(t,s)B(t,s)\times C(t,s,x)\Delta z(t,x) + \sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\psi'(t,x)\Delta_{\bar{u}(t,x)}f[t,x] +$$
$$+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\sum_{s=x_0}^{x_1}\psi'(t,s)B(t,s)\Delta_{\bar{u}(t,x)}D[t,s,x]. \tag{14}$$

Since $\Delta z(t,x) = 0$, it is easy to prove that

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\psi'(t,x)\Delta z(t+1,x) = \sum_{x=x_0}^{x_1}\psi'(t_1-1,x)\Delta z(t_1,x) +$$
$$+\sum_{t=t_0}^{t_1-1}\sum_{t=t_0}^{x_1}\psi'(t-1,x)\Delta z(t,x). \tag{15}$$

We introduce an analogue of the Hamilton-Pontryagin function in the form

$$H(t,x,u,\psi) = \psi'(t,x)f\big(t,x,u(t,x)\big) + \sum_{s=x_0}^{x_1}\psi'(t,s)B(t,s)D\big(t,s,x,u(t,x)\big),$$

and suppose that

$$\Delta_{\bar{u}(t,x)}H[t,x,u,\psi] = H\big(t,x,\bar{u}(t,x),\psi(t,x)\big) - H\big(t,x,u(t,x),\psi(t,x)\big).$$

Taking into account equalities (13)-(15) in increment formula (12), we obtain that

$$\Delta S(u) = \sum_{x=x_0}^{x_1} c'(x) z(t_1, x) + \sum_{s=x_0}^{x_1} \psi'(t_1 - 1, x) \Delta z(t_1, x) +$$
$$+ \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1} \psi'(t-1, x) \Delta z(t, x) - \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1} \psi'(t, x) A(t, x) \Delta z(t, x) -$$
$$- \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1} \psi'(t, s) B(t, s) C(t, s) \Delta z(t, x) - \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1} \Delta_{\overline{u}(t,x)} H[t, x, u, \psi]. \quad (16)$$

Assuming that $\psi(t, x)$ is the solution to the two-dimensional difference equation

$$\psi(t-1, x) = A'(t, x)\psi(t, x) + \sum_{s=x_0}^{x_1} C'(t, s, x) B'(t, s) \psi(t, s), \quad (17)$$

with the initial condition

$$\psi(t_1 - 1, x) = -c(x), \quad (18)$$

then increment formula (16) will take the form

$$\Delta S(u) = \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1} \Delta_{\overline{u}(t,x)} H(t, x, u, \psi), \quad (19)$$

Using increment formula (19) we prove

**Theorem 1.** For the optimality of the admissible control $u(t, x)$ it is necessary and sufficient in problem (1)-(4) that the inequality

$$\sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1} \Delta_{\overline{u}(t,x)} H(t, x, u, \psi) \leq 0, \quad (20)$$

hold for all admissible controls $v(t, x)$.

**Proof.**

**Necessity.** Suppose that the control $u(t, x)$ is optimal. Let us prove that relation (20) holds. By virtue of the optimality of the control $u(t, x)$ it immediately follows from (19) that inequality (20) takes place. Sufficiency also follows immediately by virtue of (20) from increment formula (19).

A direct corollary of Theorem 1 is the following statement:

**Corollary 1**. For the optimality of the admissible control $u(t, x)$ it is necessary and sufficient in problem (1)-(5) that the condition

$$\sum_{s=x_0}^{x_1} \left[ H\big(\theta, x, v(x), \psi(\theta, x)\big) - H\big(\theta, x, u(t, x), \psi(\theta, x)\big) \right] \leq 0. \quad (21)$$

hold for all $\theta = t_0, t_0 + 1, \dots, t_1 - 1, v(x) \in U, x = x_0, x_0 + 1, \dots, x_1$.

Note that the obtained optimality conditions are discrete analogues of L.S. Pontryagin's maximum condition first proved by R.V. Gamkrelidze and V.G. Boltyansky (see, e.g., [2, 5]) in the case of the optimal control problem described by a system of ordinary differential equations.

## 4. The case of a nonlinear convex functional

We consider a more general case of the quality criterion in linear control problem (2)-5).

Assume that $\phi(z)$ is a specified continuously differentiable and convex scalar function.

Suppose we need to find the minimum value of the nonlinear functional

$$\Delta S(u) = \sum_{x=x_0}^{x_1} \phi\big(z(t_1, x)\big), \tag{22}$$

under constraints (2)-(5), i.e., assuming that

$$z(t+1, x) = A(t, x)z(t, x) + B(t, x)y(t, x) + f\big(t, x, u(t, x)\big),$$
$$t = t_0, t_0 + 1, \dots, t_1 - 1; x = x_0, x_0 + 1, \dots, x_1, \tag{23}$$
$$z(t, x) = a(x), x = x_0, x_0 + 1, \dots, x_{1,} \tag{24}$$
$$\Delta y(t, x) = \sum_{x=x_0}^{x_1} [C(t, x, s)z(t, s) + D(t, x, s)u(t, s)], \tag{25}$$
$$u(t, x) \in U \subset R^r, t = t_0, t_0 + 1, \dots, t_1 - 1, x = x_0, x_0 + 1, \dots, x_1. \tag{26}$$

It is assumed that all relations (23)-(26) satisfy the smoothness conditions given in Section 2.

Suppose that $u(t, x)$ and $\bar{u}(t, x) = u(t, x) + \Delta u(t, x)$ are two admissible controls. Using an analogue of the Taylor formula, we write the increment of the quality functional in the form

$$\Delta S(u) = S(u + \Delta u) - S(u) = \sum_{x=x_0}^{x_1} \frac{\partial \phi'\big(z(t_1, x)\big)}{\partial z} \Delta z(t_1, x) + \sum_{x=x_0}^{x_1} o(\|\Delta z(t_1, x)\|). \tag{27}$$

Here, $\|\alpha\|$ means the norm of the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)'$ determined from the formula

$$\|\alpha\| = \sum_{i=1}^{\mathrm{n}} |\alpha_i|,$$

and $\mathrm{o}(\alpha)$ means that $\mathrm{o}(\alpha)/\ \alpha \to 0$.

We introduce the notation

$$M(t, x, u, p) = p'(t, x) f\big(t, x, u(t, x)\big) + \sum_{s=x_0}^{x_1} p'(t, s) B(t, s) D\big(t, s, x, u(t, x)\big),$$

where $p(t, x)$ is a $n$-dimensional vector function, which is a solution of the Fredholm difference equation

$$p(t-1, x) = A(t, x)p(t, x) + \sum_{s=x_0}^{x_1} C'(t, s, x)B(t, s)p(t, s),$$

with the initial condition

$$p(t_1 - 1, x) = -\sum_{x=x_0}^{x_1} \frac{\partial \phi\big(z(t_1, x)\big)}{\partial z}.$$

Then, by analogy with the proof of increment formula (19), increment formula (27) is presented in the form

$$\Delta S(u) = \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1} \Delta_{\bar{u}} M(t, x, u, p) + \mathrm{o}_1(\|\Delta z(t_1, x)\|). \tag{28}$$

Since the assumption is that $\phi(z)$ is a function that is convex in $z$, then by virtue of the known property of a convex differentiable function the following inequality holds:

$$\mathrm{o}_1(\|\Delta z(t_1, x)\|) \geq 0.$$

Therefore, it follows from increment formula (28) that

$$\Delta S(u) = -\sum_{x=x_0}^{x_1} \Delta_{\overline{u}} M(t, x, u, p). \qquad (29)$$

Using inequality (29), by analogy with the proof of Theorem 1 we prove

**Theorem 2**. For the optimality of the admissible control $u(t, x)$ it is sufficient in problem (22)-(26) that the inequality

$$\sum_{x=x_0}^{t_1} \sum_{x=x_0}^{x_1} \Delta_{\overline{u}} M(t, x, u, p) \leq 0$$

hold for all admissible controls $v(t, x)$.

Thus, in the case of linear right-hand side of the equation for the state vector and nonlinearity for the control we have proved sufficient optimality condition of the Pontryagin maximum principle type in the case of a nonlinear convex quality criterion.

## 5. Conclusion

In the paper, we have constructed a formula for increments of the functionals under consideration, using one variant of the method of increments. The constructed functional increment formulas allowed us to prove the optimality conditions.

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