Optimal control of continuous stochastic systems with functional inequality constraints

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ABSTRACT

We investigate the problem of optimal control of nonlinear stochastic systems whose mathematical model is described by an ordinary Ito stochastic differential equation with functional inequality constraints on the trajectory. Analogues of the stochastic Pontryagin maximum principle, the linearized maximum condition are established.

1. Introduction

As is known, stochastic dynamical systems currently occupy a significant place in the theory and practice of control [1, 2]. To date, the issue of quality control of this type of optimal control problems has been extensively discussed in the literature [3-11], etc.

In the proposed study we consider control systems, whose mathematical models are specified by nonlinear stochastic differential equations with a diffusion component, which allow taking into account the random continuous perturbations acting on the system in the presence of functional inequality constraints on the right end of the trajectory. On the basis of such systems, it is possible to simulate the behavior of quite complex objects, taking into account as continuous random impacts [9].

Note that similar deterministic systems with functional inequality constraints have been studied in [12, 13] and others.

In this work, using a generalization of the "needle-shaped variation" [12, 14], the stochastic analogues of the Pontryagin maximum principle and the linearized maximum condition [14, 15] are obtained for the stochastic problem of optimal control of systems with functional inequality constraints.

2. Problem statement

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Suppose $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t_0 \le t \le t_1}, P\right)$ is a complete probability space with a natural filter of an n-dimensional standard Wiener process $w(t), t \in T = [t_0, t_1]$. $L^2_{\mathcal{F}}(t_0, t_1; R^n)$ is the space of $\mathcal{F}^t$-consistent processes measurable in (t, ω), $x(t, \omega): T \times \Omega \to R^n$, for which $E\int_{t_0}^{t_1} \|x(t)\|^2 dt < +\infty$, where E is the symbol of mathematical expectation.

We assume that on this probability space there is a controlled process described by the Ito stochastic differential equation [1, 16]

$$dx(t) = f\big(t, x(t), u(t)\big)dt + \sigma\big(t, x(t)\big)dw(t), t \in T, \tag{1}$$

with the initial condition

$$x(t_0) = x_0. \tag{2}$$

Here $x(t) \in R^n$ is a state vector; $f(t, x, u)$ is a specified n-dimensional vector function continuous in a set of variables with partial derivatives with respect to x; $\sigma\big(t, x(t)\big): T \times R^n \to R^{n\times n}$ is an $(n \times n)$-dimensional matrix function continuous in a set of variables with partial derivatives with respect to x.

$$u(t, \omega) \in U_{ac} \equiv \{u(.,.) \in L^2_{\mathcal{F}}(t_0, t_1; R^r)/u(.,.) \in U \subset R^r, \text{п. н.}\}, \tag{3}$$

where U is a specified non-empty, bounded set (control area).

The control function $u(t) \in U_{ac}, t \in T$ will be called an admissible control, if the corresponding solution $x(t)$ of Cauchy problem (1)-(2) satisfies the constraints

$$S_i(u) = E\varphi_i\big(x(t_1)\big) \le 0, i = \overline{1, p}, \tag{4}$$

where $\varphi_i(x), i = \overline{1, p}$ specified continuously differentiable scalar functions.

It is assumed that there is a unique almost sure continuous solution $x(t)$ of problem (1)-(2) corresponding to each admissible control (t), $t \in T$.

It is required to find an admissible control $u(t)$ that affords a minimum to the objective terminal functional

$$S_0(u) = E\varphi_0\big(x(t_1)\big), \tag{5}$$

where $\varphi_0(x)$ is a specified continuously differentiable scalar function.

As noted above, the aim of the work is to establish the first-order necessary optimality conditions of the Pontryagin maximum principle type and the linearized maximum principle for the considered stochastic problem (1)-(5).

3. Functional increment formula

Consider two admissible processes: the original $\big(u(t), x(t)\big)$ and the varied $\big(\bar{u}(t) = u(t) + \Delta u(t), \bar{x}(t) = x(t) + \Delta x(t)\big)$.

Then it is clear that the problem in increments has the form:

$$d\Delta x(t) = d[\bar{x}(t) - x(t)] = (f\big(t, \bar{x}(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big))dt +$$
$$+ \Big(\sigma\big(t, \bar{x}(t)\big) - \sigma\big(t, x(t)\big)\Big) dw(t),$$
$$\Delta x(t_0) = 0. \tag{6}$$

Suppose $\psi_i(t) \in L^2_F(t_0, t_1; R^n)$, $\beta_i(t) \in L^2_F(t_0, t_1; R^{n\times n})$, $i = \overline{0, p}$ is a is a random process, the stochastic differential of which has the form:

$$d\psi_i(t) = \alpha_i(t)dt + \beta_i(t)dw(t), i = \overline{0, p}.$$

Here $\alpha_i(t), i = \overline{0,p}$ is n-dimensional measurable and bounded functions.
Hence, based on Ito's formula [1, 16], the following identity takes place:

$$d\left(\psi_i{}'(t)\Delta x(t)\right) = d\psi_i{}'(t)\Delta x(t) +$$
$$+\psi_i'(t)\left[f\left(t,\bar{x}(t),\bar{u}(t)\right) - f\left(t,x(t),u(t)\right)\right]dt +$$
$$+\psi_i{}'(t)\left(\sigma\left(t,\bar{x}(t)\right) - \sigma\left(t,x(t)\right)\right)dw(t) + \beta_i(t)\left(\sigma\left(t,\bar{x}(t)\right) - \sigma\left(t,x(t)\right)\right)dt. \qquad (7)$$

Here and further, $(')$ is the transposition sign.

Suppose $I(u) = \left\{i: E\varphi_i\left(x(t_1)\right) = 0, i = \overline{1,p}\right\}, J(u) = \{0\} \cup I(u)$.

In order to simplify the calculations, we will assume that in problem (1)-(5),

$$I(u) = \{1,2, \dots, m\}, m \le p.$$

We introduce the notation:

$$H^{(i)}(t,x,u,\psi_i) = \psi_i'(t)f(t,x,u), H_x^{(i)}[t] = H_x\left(t,x(t),u(t),\psi_i(t)\right), i \in J(u),$$
$$\Delta_{\bar{u}(t)}H^{(i)}[t] = H^{(i)}\left(t,x(t),\bar{u}(t),\psi_i(t)\right) - H^{(i)}\left(t,x(t),u(t),\psi_i(t)\right),$$
$$\sigma_x[t] = \sigma_x\left(t,x(t)\right).$$

Now we will require that the random processes $\psi_i(t) \in L_{\mathcal{F}}^2(t_0,t_1;R^n)$ and $\beta_i(t) \in L_{\mathcal{F}}^2(t_0,t_1;R^{n\times n})$ are solutions of the following system of stochastic differential equations (conjugate system):

$$\begin{cases} d\psi_i(t) = -\left(H_x^{(i)}[t] + \beta_i(t)\sigma_x[t]\right)dt + \beta_i(t)dw(t), \\ \psi_i(t_1) = -\dfrac{\partial\varphi_i\left(x(t_1)\right)}{\partial x}, i \in J(u). \end{cases}$$

Taking this into account, by similar schemes from works [6-8] the functional increments can be represented in the form:

$$\Delta S_i(u) = S_i(u+\Delta u) - S_i(u) = -E\int_{t_0}^{t_1}\Delta_{\bar{u}}H^{(i)}[t]dt + \eta_1^{(i)}(t,\Delta u(t)). \qquad (8)$$

Here,

$$\eta_1^{(i)}(t,u(t)) = E\left\{o_1{}^{(i)}(\|\Delta x(t_1)\|) - \int_{t_0}^{t_1} o_2{}^{(i)}(\|\Delta x(t)\|)dt - \int_{t_0}^{t_1}\Delta_{\bar{u}}H_x'[t]\Delta x(t)dt\right\}, \qquad (9)$$

is the residual term, and the quantities $o_1(\|\Delta x(t_1)\|)$ and $o_2(\|\Delta x(t)\|)$ are determined from the corresponding expansions:

$$\varphi_i\left(\bar{x}(t_1)\right) - \varphi_i\left(x(t_1)\right) = \frac{\partial\varphi_i(t_1)}{\partial x}\Delta x(t_1) + o_1(\|\Delta x(t_1)\|),$$
$$H^{(i)}\left(t,\bar{x}(t),\bar{u}(t),\psi_i(t)\right) - H^{(i)}\left(t,x(t),\bar{u}(t),\psi_i(t)\right) =$$
$$= H_x{}^{(i)'}\left(t,x(t),\bar{u}(t),\psi_i(t)\right)\Delta x(t) + o_2(\|\Delta x(t)\|), i \in J(u),$$

where $\|a\|$ is the norm of the vector $a = (a_1,a_2,\dots,a_n)'$ determined from the formula

$$\|a\| = \sum_{i=1}^{n}|a_i|,$$

and $o(\alpha)$ is a quantity of a higher order than α, i.e., $\frac{o(\alpha)}{\alpha} \to 0$ at $\alpha \to 0$.

4. Necessary optimality condition of the Pontryagin maximum principle type

This paragraph proves the following assertion.

Theorem 1. It is necessary for the optimality of the admissible control $u(t), t \in T$ in stochastic problem (1)-(5) that the inequality

$$\min_{i \in J(u)} E \sum_{j=1}^{m+1} l_j \left[H^{(i)}\left(\theta_j, x(\theta_j), v_j, \psi_i(\theta_j)\right) - H^{(i)}\left(\theta_j, x(\theta_j), u(\theta_j), \psi_i(\theta_j)\right) \right] \leq 0, \tag{10}$$

hold for all $v_j \in U, l_j \geq 0, \theta_j \in [t_0, t_1), j = \overline{1, m+1}$.

In the proof of Theorem 1 we will use the method of proof by contradiction. Suppose it is not true, i.e., the control $u(t), t \in T$ is optimal, but $\bar{l}_j \geq 0, \bar{v}_j \in U, \bar{\theta}_j \in\in [t_0, t_1)$ $(t_0 \leq \bar{\theta}_1 \leq \bar{\theta}_2 \leq \cdots \bar{\theta}_{m+1} < t_1)$ exist such that

$$\min_{i \in J(u)} E \sum_{j=1}^{m+1} \bar{l}_j \left[H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), \bar{v}_j, \psi_i(\bar{\theta}_j)\right) - H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), u(\bar{\theta}_j), \psi_i(\bar{\theta}_j)\right) \right] > 0. \tag{11}$$

The special increment of the optimal control $u(t)$ is determined from the formula

$$\Delta \bar{u}_\varepsilon(t) = \sum_{j=1}^{\bar{m}+1} \delta u\left(t, \varepsilon; \bar{\theta}_j, \bar{l}_j, \bar{v}_j\right). \tag{12}$$

Here $\delta u\left(t, \varepsilon; \bar{\theta}_j, \bar{l}_j, \bar{v}_j\right)$ is a needle-shaped variation of the control:

$$\delta u\left(t, \varepsilon; \bar{\theta}_j, \bar{l}_j, \bar{v}_j\right) = \begin{cases} \bar{v}_j - u(t), t \in [\bar{\theta}_j, \bar{\theta}_j + \bar{l}_j \varepsilon) \\ 0, \qquad t \in T \setminus \left[\bar{\theta}_j, \bar{\theta}_j + \bar{l}_j \varepsilon\right). \end{cases}$$

Note that the summation of needle-shaped variations is understood as in the works [12, 13].

By reasoning similar to [7-9], it is easy to prove that the norm of the special increment $\|\Delta x_\varepsilon(t)\|, t \in T$ of the trajectory $x(t)$, corresponding to increment (12) of the control $u(t), t \in T$ has an order of smallness ε, and the residual term determined from formula (9) has an order of smallness $o(\varepsilon)$, i.e.,

$$\eta_1^{(i)}(t, \Delta u_\varepsilon(t)) = o(\varepsilon). \tag{13}$$

Hence, taking into account (12) and this fact we obtain from (8) the validity of the expansions

$$S_i(u + \Delta \bar{u}_\varepsilon) - S_i(u) =$$

$$= -\varepsilon E \sum_{j=1}^{m+1} \bar{l}_j \left[H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), \bar{v}_j, \psi_i(\bar{\theta}_j)\right) - H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), u(\bar{\theta}_j), \psi_i(\bar{\theta}_j)\right) \right] + o(\varepsilon). \tag{14}$$

Consider the case of $i \in I(u)$. Then, taking into account (12), we obtain from expansion (14) that

$$S_i(u + \Delta \bar{u}_\varepsilon) =$$

$$= -\varepsilon E \sum_{j=1}^{\bar{m}+1} \bar{l}_j \left[H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), \bar{v}_j, \psi_i(\bar{\theta}_j)\right) - H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), u(\bar{\theta}_j), \psi_i(\bar{\theta}_j)\right) \right] + o(\varepsilon) < 0.$$

And at $i \in \{\overline{1, p}\} \setminus I(u)$, due to the continuity of the functions $\varphi_i(x)$, it follows that

$$S_i(u + \Delta \bar{u}_\varepsilon) = E\varphi(x(t_1)) -$$

$$-\varepsilon E \sum_{j=1}^{m+1} \bar{l}_j \left[H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), \bar{v}_j, \psi_i(\bar{\theta}_j)\right) - H^{(i)}\left(\bar{\theta}_j, x(\bar{\theta}_j), u(\bar{\theta}_j), \psi_i(\bar{\theta}_j)\right) \right] + o(\varepsilon) < 0.$$

These relations show that the "varied" control $\bar{u}_\varepsilon(t) = u(t) + \Delta u_\varepsilon(t)$ is an admissible control.

But at the same time, from expansion (14), taking into account (11), it follows that

$$S_0(u + \Delta\bar{u}_\varepsilon) < S_0(u)$$

which means the suboptimality of the control $u(t)$. Consequently, we have obtained a contradiction. This theorem is proved.

5. Linearized necessary optimality condition

Now consider problem (1)-(5) under the assumption that the following conditions are met:

a) U is a specified non-empty, bounded and convex set;

b) $f(t, x, u)$ is continuous in a set of variables with partial derivatives with respect to (x, u).

Note that if conditions (a), (b) are satisfied, the formula for the functional increment will take the form (see e.g. [7-9])

$$\Delta S_i(u) = S_i(u + \Delta u) - S_i(u) =$$

$$= -E\int_{t_0}^{t_1} H_u^{(i)'}\big(t, x(t), u(t), \psi_i(t)\big)\Delta u(t)dt + \eta_2^{(i)}\big(t, \Delta u(t)\big), i = \overline{0,p} \quad (15)$$

where the residual term $\eta_2^{(i)}\big(t, \Delta u(t)\big)$ is determined from the formula

$$\eta_2(t, u(t)) = E\left\{o_1^{(i)}(\|\Delta x(t_1)\|) - \int_{t_0}^{t_1} o_3^{(i)}(\|x(t) + u(t)\|)dt\right\}.$$

Since the set U is convex, the special increment of the control $u(t)$ can be determined from the formula

$$\Delta u(t; \mu) = \mu\big(v(t) - u(t)\big), t \in T,$$

where $0 \le \mu \le 1$, and $v(t) \in U, t \in [t_0, t_1]$ is an arbitrary admissible control.

Assume that $u(t), t \in T$ is an optimal control and $v(t)$ is determined as follows

$$v_\mu(t) = \sum_{j=1}^{m+1} \delta u\big(t, \mu; \theta_j, l_j, v_j\big), \quad (16)$$

where $\delta u\big(t, \mu; \theta_j, l_j, v_j\big)$ is determined from the formula

$$\delta u\big(t, \mu; \theta_j, l_j, v_j\big) = \begin{cases} v_i, & t \in [\theta_j, \theta_j + l_j\mu) \\ u(t), & t \in T \setminus \big[\theta_j, \theta_j + l_j\mu\big). \end{cases}$$

Taking into account (16) in expansion (15) by reasoning similar to the proof of Theorem 1, we arrive at the following assertion.

Theorem 2. Suppose that the set U is convex, and $f(t, x, u)$ is continuous in a set of variables with partial derivatives with respect to (x, u). Then it is necessary for the optimality of the admissible control $u(t), t \in T$ in stochastic problem (1)-(5) that the inequality

$$\min_{i \in J(u)} E\sum_{j=1}^{m+1} l_j {H_u}^{(i)}\left(\theta_j, x(\theta_j), u(\theta_j), \psi_i(\theta_j)\right)\left(v_j - u(\theta_j)\right) \le 0,$$

hold for all $v_j \in U, l_j \ge 0, \theta_j \in [t_0, t_1), j = \overline{1, m+1}$.

6. Conclusion

The stochastic problem of optimal control in the presence of functional inequality constraints on the right end of the trajectory is considered. Under different assumptions the first-order necessary optimality conditions for the problem under investigation are established.

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