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Theoretical substantiation and experimental analysis of the method for finding the upper bound for the interval problem of mixed-Boolean programming

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ABSTRACT

There are various methods for finding an approximate solution to a mixed-Boolean programming problem. Obviously, after finding such approximate solutions that corresponding to the value of the objective function, it is necessary to estimate the proximity of this value to the optimal one. For this, the corresponding linear programming problems are usually solved (i.e., the integer condition is discarded). This approach can be ineffective in the case of large dimensions, since the coordinates of the optimal solution and the optimal value of the objective function are determined. Obviously, to find the absolute and relative errors, there is no need to find the coordinates of the optimal solution, however, the approximate values of the functional and its upper bound are needed. For this purpose, in this paper, an algorithm for finding the upper bound of the interval problem of mixed-Boolean programming has been developed. For this, the majorizing function of the Lagrange-type for this problem is minimized.

1. Introduction

The following problem is considered:

$$\sum_{j=1}^{n}\left[\underline{c}_j,\overline{c}_j\right]x_j+\sum_{j=n+1}^{N}\left[\underline{c}_j,\overline{c}_j\right]x_j\rightarrow \max \tag{1}$$

$$\sum_{j=1}^{n}\left[\underline{a}_{ij},\overline{a}_{ij}\right]x_j+\sum_{j=n+1}^{N}\left[\underline{a}_{ij},\overline{a}_{ij}\right]x_j\leq\left[\underline{b}_i,\overline{b}_i\right],\quad \left(i=\overline{1,m}\right) \tag{2}$$

$$0\leq x_j\leq 1,\ \left(j=\overline{1,N}\right), \tag{3}$$

$$x_j=1\vee 0,\ \left(j=\overline{1,n}\right),\ (n\leq N) \tag{4}$$

Here $0<\underline{c}_j\leq\ \overline{c}_j, 0\leq\underline{a}_{ij}\leq\ \overline{a}_{ij}, 0<\underline{b}_i\leq\overline{b}_i, \left(i=\overline{1,m};\ j=\overline{1,N}\right)$ are given integers.

For this problem, using the methods developed in [1], an approximate value of function (1) is

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found. In order to estimate the proximity of this value to the optimal one, the following function was constructed in [2], based on the optimistic and pessimistic strategies.

$$L^{op}(\lambda_1,\lambda_2,\dots,\lambda_m)=\sum_{j\in\omega_1^{op}}\overline{c}_j+\sum_{j\in\omega_2^{op}}\overline{c}_j+\sum_{i=1}^{m}\left(b_i-\sum_{j\in\omega_1^{op}}\underline{a}_{ij}-\sum_{j\in\omega_2^{op}}\underline{a}_{ij}\right)\lambda_i, \tag{5}$$

$$L^{pes}(\lambda_1,\lambda_2,\dots,\lambda_m)=\sum_{j\in\omega_1^{op}}\underline{c}_j+\sum_{j\in\omega_2^{op}}\underline{c}_j+\sum_{i=1}^{m}\left(b_i-\sum_{j\in\omega_1^{op}}\overline{a}_{ij}-\sum_{j\in\omega_2^{op}}\overline{a}_{ij}\right)\lambda_i, \tag{6}$$

$$\left.\begin{aligned}\omega_1^{op}&=\left\{1\le j\le n\middle|\overline{c}_j-\textstyle\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i>0\right\},\\ \omega_2^{op}&=\left\{n+1\le j\le N\middle|\overline{c}_j-\textstyle\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i>0\right\},\end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned}\omega_1^{pes}&=\left\{1\le j\le n\middle|\underline{c}_j-\textstyle\sum_{i=1}^{m}\overline{a}_{ij}\lambda_i>0\right\},\\ \omega_2^{pes}&=\left\{n+1\le j\le N\middle|\underline{c}_j-\textstyle\sum_{i=1}^{m}\overline{a}_{ij}\lambda_i>0\right\},\end{aligned}\right\} \tag{8}$$

In this paper, an algorithm for minimizing functions (5), (6) is developed, as a result of which an upper bound for the optimal value is obtained.

2. Problem statement

Problem (1)-(4) is called a mixed-Boolean programming problem with interval data. In [1], a method was developed for finding an approximate solution, and in [2], an algorithm was proposed for finding the upper limit of the optimum for this problem. Based on different strategies, in [3-8] special methods of approximate solution for an integer or mixed integer interval problem were developed.

Note that for problem (1)-(4), the authors of this work have developed methods for constructing approximate (suboptimistic and subpessimistic) solutions and published in [7-8]. It is clear that after finding an approximate solution, it is necessary to estimate the errors of the found solution from the optimal one. For this, linear programming problems are usually solved, discarding the integer condition of (4), and choosing the coefficients in a special way.

It should be noted that when repeatedly solving a linear programming problem of large dimension one can encounter serious difficulties. Therefore, there is a need to find the upper bound of the optimal value faster than using the linear programming apparatus. For this purpose, in this work, we have developed such an algorithm for finding the upper bound.

Note that the interval linear programming problem was investigated in [9-11], that is, without conditions that the variables are integer.

3. Theoretical substantiation of the method

In the authors' paper [3], Lagrange-Type functions (5)-(6) were constructed for the optimistic and pessimistic problems, respectively, and the following theorems were proved:

Theorem 1. For the optimistic value f_{op}^* and pessimistic value f_{pes}^* of the objective function of problem (1) - (4), the following inequalities are valid:

$$f_{op}^*\le\min_{\lambda_i\ge0}L^{op}(\lambda_1,\lambda_2,\dots,\lambda_m),$$

$$f_{pes}^*\le\min_{\lambda_i\ge0}L^{pes}(\lambda_1,\lambda_2,\dots,\lambda_m).$$

Theorem 2. The functions $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $L^{pes}(\lambda_1, \lambda_2, \dots, \lambda_m)$ are piecewise linear, continuous, non-differentiable, and convex.

Proof. First, we prove these theorems for the function $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$. The proof of this theorem for the function $L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m)$ is similar. We write the function $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ in the following form:

$$L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m) = C(\omega_1^{op}) + C(\omega_2^{op}) + \\ + \sum_{i=1}^{m} \left(b_i - A_i(\omega_1^{op}) - A_i(\omega_2^{op})\right) \lambda_i, \quad (9)$$

где

$$C(\omega_1^{op}) = \sum_{j \in \omega_1^{op}} \bar{c}_j, \qquad C(\omega_2^{op}) = \sum_{j \in \omega_2^{op}} \bar{c}_j,$$

$$A_i(\omega_1^{op}) = \sum_{j \in \omega_1^{op}} \underline{a}_{ij}, \qquad A_i(\omega_2^{op}) = \sum_{j \in \omega_2^{op}} \underline{a}_{ij}, \qquad (i = \overline{1, m}).$$

Note that by changing the sets ω_1 and ω_2, the coefficients (ω_1^{op}), $C(\omega_2^{op})$, $A_i(\omega_1^{op})$ and $A_i(\omega_2^{op})$, $(i = \overline{1, m})$ of the function (9) change. And this means that the linear function (9) changes its direction and turns into another linear function. Obviously, in this case it remains continuous, since all arguments $\lambda_i \geq 0$, $(i = \overline{1, m})$ of this function change continuously. Hence, a piecewise linear and continuous function is obtained. It is clear that the piecewise linear function does not have a unique derivative at the breaking point. Therefore, the function $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ is not differentiable.

To prove the convexity of the function $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ we write it in the following form:

$$L^{op}(\Lambda) = \sum_{i=1}^{m} b_i \lambda_i + \sum_{j=1}^{n} max\left\{0; \bar{c}_j - \sum_{i=1}^{m} \underline{a}_{ij} \lambda_i\right\} \\ + \sum_{j=n+1}^{N} max\left\{0; \bar{c}_j - \sum_{i=1}^{m} \underline{a}_{ij} \lambda_i\right\}, \quad (10)$$

where $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$.

To prove the convexity of $L^{op}(\Lambda)$ we show that the well-known convexity condition is satisfied:

$$L^{op}(t\Lambda' + (1-t)\Lambda'') \leq tL^{op}(\Lambda') + (1-t)L^{op}(\Lambda'')$$

or

$$tL^{op}(\Lambda') + (1-t)L^{op}(\Lambda'') - L^{op}(t\Lambda' + (1-t)\Lambda'') \geq 0 \quad (11)$$

Here $\Lambda' \geq 0, \Lambda'' \geq 0$ и $0 \leq t \leq 1$.

Substituting (10) into (11), we obtain

$$t \sum_{i=1}^{m} b_i \lambda_i' + t \sum_{j=1}^{n} max\left\{0; \bar{c}_j - \sum_{i=1}^{m} \underline{a}_{ij} \lambda_i'\right\} + t \sum_{j=n+1}^{N} max\left\{0; \bar{c}_j - \sum_{i=1}^{m} \underline{a}_{ij} \lambda_i'\right\} + \\ +(1-t) \sum_{i=1}^{m} b_i \lambda_i'' + (1-t) \sum_{j=1}^{n} max\left\{0; \bar{c}_j - \sum_{i=1}^{m} \underline{a}_{ij} \lambda_i''\right\} + \\ +(1-t) \sum_{j=n+1}^{N} max\left\{0; \bar{c}_j - \sum_{i=1}^{m} \underline{a}_{ij} \lambda_i''\right\} -$$

$$-\sum_{i=1}^{m} b_i(t\lambda_i' + (1-t)\lambda_i'') - \sum_{j=1}^{n} max\left\{0; \overline{c}_j - \sum_{i=1}^{m} \underline{a}_{ij}(t\lambda_i' + (1-t)\lambda_i'')\right\} -$$
$$-\sum_{j=n+1}^{N} max\left\{0; \overline{c}_j - \sum_{i=1}^{m} \underline{a}_{ij}(t\lambda_i' + (1-t)\lambda_i'')\right\} = \sum_{j=1}^{n} max\left\{0; t\overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i'\right\} +$$
$$+\sum_{j=1}^{n} max\left\{0; (1-t)\overline{c}_j - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\} + \sum_{j=n+1}^{N} max\left\{0; t\overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i'\right\} +$$
$$+\sum_{j=n+1}^{N} max\left\{0; (1-t)\overline{c}_j - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\} -$$
$$-\sum_{j=1}^{n} max\left\{0;\ \overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\} -$$
$$-\sum_{j=n+1}^{N} max\left\{0;\ \overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\} =$$
$$= \sum_{j=1}^{n} \delta_j' + \sum_{j=1}^{n} \delta_j'' + \sum_{j=n+1}^{N} \delta_j''' + \sum_{j=n+1}^{N} \delta_j^{IV} - \sum_{j=1}^{n} \delta_j^{V} - \sum_{j=n+1}^{N} \delta_j^{VI} =$$
$$= \sum_{j=1}^{n}\left(\delta_j' + \delta_j'' - \delta_j^{V}\right) + \sum_{j=n+1}^{N}\left(\delta_j''' + \delta_j^{IV} - \delta_j^{VI}\right) = \sum_{j=1}^{n} \Delta_j^1 + \sum_{j=n+1}^{N} \Delta_j^2.$$

Здесь

$$\Delta_j^1 = \delta_j' + \delta_j'' - \delta_j^{V},\ (j = \overline{1,n}),$$
$$\Delta_j^2 = \delta_j''' + \delta_j^{IV} - \delta_j^{VI},\ (j = \overline{n+1,N}).$$
$$\delta_j' = \sum_{j=1}^{n} max\left\{0; t\overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i'\right\},$$
$$\delta_j'' = \sum_{j=1}^{n} max\left\{0; (1-t)\overline{c}_j - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\},$$
$$\delta_j^{V} = \sum_{j=1}^{n} max\left\{0;\ \overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\},$$
$$\delta_j^{IV} = \sum_{j=n+1}^{N} max\left\{0; t\overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i'\right\},$$
$$\delta_j''' = \sum_{j=n+1}^{N} max\left\{0; (1-t)\overline{c}_j - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\},$$
$$\delta_j^{VI} = \sum_{j=n+1}^{N} max\left\{0;\ \overline{c}_j - t\sum_{i=1}^{m} \underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m} \underline{a}_{ij}\lambda_i''\right\}.$$

Now let's prove that

$$\sum_{j=1}^{n} \Delta_j^1 + \sum_{j=n+1}^{N} \Delta_j^2 \geq 0.$$

To do this, it is necessary to show the validity of the relations:
$\Delta_j^1 \geq 0, (j = \overline{1,n})$ и $\Delta_j^2 \geq 0$, $(j = \overline{n+1,N})$. Let us first show that the inequality is fair $\Delta_j^1 \geq 0, (j = \overline{1,n})$.

There are 4 possible cases:
1)$\delta_j' > 0, \delta_j'' > 0, (j = \overline{1,n})$.
2)$\delta_j' > 0, \delta_j'' = 0, (j = \overline{1,n})$.
3)$\delta_j' = 0, \delta_j'' > 0, (j = \overline{1,n})$.
4)$\delta_j' = 0, \delta_j'' = 0, (j = \overline{1,n})$.
Let's consider all these cases separately:

1) $$\delta_j' + \delta_j'' = t\overline{c}_j - t\sum_{i=1}^{m}\underline{a}_{ij}\,\lambda_i' + (1-t)\overline{c}_j - (1-t)\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i'' =$$
$$= \overline{c}_j - t\sum_{i=1}^{m}\underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i'' = \delta_j^V, (j = \overline{1,n}).$$

Thus, $\Delta_j^1 = \delta_j' + \delta_j'' - \delta_j^V = 0, (j = \overline{1,n})$.
2) Let $\delta_j' > 0, \delta_j'' = 0, (j = \overline{1,n})$.
Then,
$$\delta_j' + \delta_j'' = t\overline{c}_j - t\sum_{i=1}^{m}\underline{a}_{ij}\,\lambda_i' + (1-t)\overline{c}_j - (1-t)\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i'' =$$
$$= \overline{c}_j - t\sum_{i=1}^{m}\underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i''.$$

Takin into account $\delta_j' > 0$, and $\delta_j'' = 0, (j = \overline{1,n})$, we obtain:
$$\delta_j' + \delta_j'' \geq max\left\{0; \overline{c}_j - t\sum_{i=1}^{m}\underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i''\right\} = \delta_j^V.$$

So, it turns out
$\Delta_j^1 = \delta_j' + \delta_j'' - \delta_j^V \geq 0$, $(j = \overline{1,n})$.

Note that the case 3 is proved similarly.
4) Let $\delta_j' = 0, \delta_j'' = 0, (j = \overline{1,n})$. It means that
$$t\overline{c}_j - t\sum_{i=1}^{m}\underline{a}_{ij}\,\lambda_i' \leq 0, (j = \overline{1,n}) \text{ и } (1-t)\overline{c}_j - (1-t)\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i'' \leq 0, (j = \overline{1,n}).$$

Adding these inequalities in parts, we get:
$$\overline{c}_j - t\sum_{i=1}^{m}\underline{a}_{ij}\,\lambda_i' - (1-t)\sum_{i=1}^{m}\underline{a}_{ij}\lambda_i'' \leq 0.$$

Therefore, $\delta_j^V = 0$, $(j = \overline{1,n})$.

Then $\Delta_j^1 = \delta_j' + \delta_j'' - \delta_j^V = 0, (j = \overline{1,n})$.

It should be noted that the proof of the inequality $\Delta_j^2 \geq 0$, $(j = \overline{n+1,N})$ is carried out in exactly the same way. Thus, the convexity of the function $L^{op}(\Lambda)$ is proved. The proof of piecewise linearity, continuity, non-differentiability and convexity of the function $L^p(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is carried out similarly.

It can be seen from these theorems that the functions $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $L^{pes}(\lambda_1, \lambda_2, \dots, \lambda_m)$ have unique minima, and this creates the mathematical foundations for minimizing these functions.

First, we consider the algorithm for minimizing a function (5).

At the beginning of the process of minimizing function (5), we accept $\lambda_i := 0,\ (i = \overline{1,m})$.

Then, taking this into account in (7), we obtain $\omega_1^{op} = \{1,2,\dots,n\}$, $\omega_2^{op} = \{n+1, n+ +2, \dots, N\}$, $\omega_1^{pes} = \{1,2,\dots,n\}$, $\omega_2^{pes} = \{n+1, n+2, \dots, N\}$.

Now we write function (5) in the following form:

$$L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m) = C(\omega_1^{op}) + C(\omega_2^{op}) + \sum_{i=1}^{m} (b_i - A_i(\omega_1^{op}) - A_i(\omega_2^{op}))\lambda_i, \tag{12}$$

where

$$C(\omega_1^{op}) = \sum_{j\in\omega_1^{op}} \overline{c}_j,\ C(\omega_2^{op}) = \sum_{j\in\omega_2^{op}} \overline{c}_j,$$

$$A_i(\omega_1^{op}) = \sum_{j\in\omega_1^{op}} \underline{a}_{ij}, \qquad A_i(\omega_2^{op}) = \sum_{j\in\omega_2^{op}} \underline{a}_{ij},\ \ (i = \overline{1,m}).$$

Obviously, for each $i, (i = \overline{1,m})$, the relation $\left(b_i - A_i(\omega_1^{op}) - A_i(\omega_2^{op})\right) < 0$ must be fulfilled.

If for some number i this relation is not fulfilled, then the corresponding inequality i will not be a restriction. For simplicity of calculations, we write the function $L^{op}(\Lambda)$ in the following form:

$$L^{op}(\Lambda) = C(\omega_1^{op}, \omega_2^{op}) + \sum_{i=1}^{m} \left(B_i(\omega_1^{op}, \omega_2^{op})\right)\lambda_i. \tag{13}$$

Here $C(\omega_1^{op}, \omega_2^{op}) = C(\omega_1^{op}) + C(\omega_2^{op})$, $B_i(\omega_1^{op}, \omega_2^{op}) = b_i - A_i(\omega_1^{op}) - A_i(\omega_2^{op})$, $(i = \overline{1,m})$.

It is clear that if for all $i,\ (i = \overline{1,m})$. $B_i(\omega_1^{op}, \omega_2^{op}) \geq 0$, then the process of minimization is completed, since further increase in parameters $\lambda_i, (i = \overline{1,m})$ will increase the value of function (13). And this requires the existence of a negative coefficient $B_{i_*}(\omega_1^{op}, \omega_2^{op})$ at λ_{i_*}.

To speed up minimization, it should be choose a number i_* from the following relation:

$$\min_i \left(B_i(\omega_1^{op}, \omega_2^{op})\right) = B_{i_*}(\omega_1^{op}, \omega_2^{op}).$$

In other words,

$$i_* = arg\left(\min_i \left(B_i(\omega_1^{op}, \omega_2^{op})\right)\right). \tag{14}$$

It should be noted that to determine the valueλ_{i_*}, it is necessary to take into account the structure of the sets ω_1^{op} and ω_2^{op}. For this, we write the relation (7) in the following form:

$$\overline{c}_j - \sum_{i\neq i_0} \underline{a}_{ij}\lambda_i - \underline{a}_{i_*j}\lambda_{i_*} > 0, \qquad \left(j \in \omega_1^{op} \cup \omega_2^{op}\right).$$

From here

$$\lambda_{i_*} < (\overline{c}_j - \sum_{i\neq i_0} \underline{a}_{ij}\lambda_i)/\underline{a}_{i_*j}, \qquad \left(j \in \omega_1^{op} \cup \omega_2^{op}\right). \tag{15}$$

Obviously, for λ_{i_*} it is necessary to take the smallest value from relations (15). Otherwise, several elements are simultaneously excluded from the set $\omega_1^{op} \cup \omega_2^{op}$.

Then

$$\lambda_{i_*} = \min_{j\in\omega_1^{op}\cup\omega_2^{op}} \left\{ \frac{\overline{c}_j - \sum_{i\neq i_0} \underline{a}_{ij}\lambda_i}{\underline{a}_{i_*j}} \right\} = \frac{\overline{c}_{j_*} - \sum_{i\neq i_0} \underline{a}_{ij_*}\lambda_i}{\underline{a}_{i_*j_*}}, \qquad \left(j \in \omega_1^{op} \cup \omega_2^{op}\right). \qquad (16)$$

It is clear that an element j_* is excluded from the sets $j \in \omega_1^{op} \cup \omega_2^{op}$.

Then $\omega_1^{op} \cup \omega_2^{op} := \omega_1^{op} \cup \omega_2^{op} \setminus \{j_*\}$. Since the sets ω_1^{op} and ω_2^{op} are changing, the function (13) also changes. In this case, the coefficients of function (13) will take on the following new values:

$$C\left(\omega_1^{op}, \omega_2^{op}\right) := C\left(\omega_1^{op}, \omega_2^{op}\right) - \overline{c}_{j_*},$$

$$B_i\left(\omega_1^{op}, \omega_2^{op}\right) := B_i\left(\omega_1^{op}, \omega_2^{op}\right) + \underline{a}_{ij_*}, \qquad (i = \overline{1,m}).$$

This ends one step of the process of minimization of the function (13). To continue the calculations, it is necessary to find new values i_* from (14) and j_* from (16). This process of minimization is completed when, for any $i, (i = \overline{1,m})$, the relation $B_i\left(\omega_1^{op}, \omega_2^{op}\right) \geq 0$ is fulfilled.

It should be noted that after each step, the values $B_i(\omega_1^{op}, \omega_2^{op})$, $(i = \overline{1,m})$ increase, since positive integers $\underline{a}_{ij_*}$, $(i = \overline{1,m})$ are added here.

Thus, this shows that after a finite number of steps, the above minimization process is completed.

Note that to minimize the function $L^{pes}(\lambda_1, \lambda_2, \dots, \lambda_m)$ written in formula (6), the minimization process is carried out entirely analogously the same way as for the function $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$.

Now let's write an algorithm for this method to minimize the function $L^{op}(\Lambda)$.

Algorithm.

Step 1. Input m, n, N. $\overline{c}_j$, $\underline{a}_{ij}$, b_i, $\left(i = \overline{1,m};\ j = \overline{1,N}\right)$.

Step 2. Accept $\omega_1^{op} = \{1,2,\dots,n\}$, $\omega_2^{op} = \{n+1, n+2, \dots, N\}$,

$$L^{op} := \sum_{j\in\omega_1^{op}\cup\omega_2^{op}} \overline{c}_j,$$

Step 3. $S := \sum_{j\in\omega_1^{op}\cup\omega_2^{op}} \overline{c}_j$, $R_i := b_i - \sum_{j\in\omega_1^{op}\cup\omega_2^{op}} \underline{a}_{ij}$, $(i = \overline{1,m})$.

Step 4. Find $\min_i R_i = R_{i_*}$.

Step 5. If $R_{i_*} \geq 0$, pass to step 11.

Step 6. Compute $\lambda_{i_*} := \min_{j\in\omega_1^{op}\cup\omega_2^{op}} \left\{ (\overline{c}_j - \sum_{i\neq i_0} \underline{a}_{ij}\lambda_i)/\underline{a}_{i_*j} \right\} = (\overline{c}_{j_*} - \sum_{i\neq i_0} \underline{a}_{ij_*}\lambda_i)/\underline{a}_{i_*j_*}$,

for $\underline{a}_{i_*j} > 0$, $(j \in \omega_1^{op} \cup \omega_2^{op})$.

Step 7. Accept $\omega_1^{op} \cup \omega_2^{op} := \omega_1^{op} \cup \omega_2^{op} \setminus \{j_*\}$.

Step 8. Accept $S := S - \overline{c}_{j_*}$, $R_i = R_{i_*} + \underline{a}_{ij_*}$, $(i = \overline{1,m})$.

Step 9. Compute $L^{op} := S + \sum_{i=1}^{m} R_i\lambda_i$.

Step 10. Pass to step 5.

Step 11. Print $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and λ_i, $(i = \overline{1,m})$.

Step 12. Stop.

4. Results of the computational experiments

To clarify the proximity of the upper bounds obtained by the linear programming method and the method developed in this work, numerous computational experiments were carried out. In the experiments, the coefficients of the problems were chosen as pseudo-random numbers that satisfy the relations:

$$0 \le \underline{a}_{ij} \le 99, \ \ 1 \le \overline{a}_{ij} \le 99, \ \ \ 1 \le \underline{c}_j \le 99, \ \ \ 1 \le \overline{c}_j \le 99, \left(i = \overline{1,m}; j = \overline{1,N}\right).$$

$$0 \le \underline{a}_{ij} \le 999, \ \ 1 \le \overline{a}_{ij} \le 999, \ 1 \le \underline{c}_j \le 999, \ \ 1 \le \overline{c}_j \le 999, \left(i = \overline{1,m}; j = \overline{1,N}\right).$$

$$\underline{b}_i := \left[\frac{1}{3}\sum_{j=1}^{N} \underline{a}_{ij}\right], \qquad \overline{b}_i := \left[\frac{1}{3}\sum_{j=1}^{N} \overline{a}_{ij}\right], \qquad \left(i = \overline{1,m}; j = \overline{1,N}\right).$$

The relative errors are estimated as follows:

$$\delta_{op} = \frac{\bar{f}_{op}^{d} - \overline{f}_{op}}{\bar{f}_{op}^{d}} \times 100\%, \ \ \delta_{pes} = \frac{\bar{f}_{pes}^{d} - \overline{f}_{pes}}{\bar{f}_{pes}^{d}} \times 100\%.$$

In the tables, the following designations are accepted:
$N -$ number of all variables;
$n -$ number of booleans;
$m -$ number of restrictions;
$\overline{f}^{op}$ and $\overline{f}^{p}$ the upper bounds of the optimistic and pessimistic values of the functional in the problem (1)-(4), respectively, using the linear programming apparatus;
$\overline{f}_d^{op}$ and $\overline{f}_d^{p}$ the upper bounds of the optimistic and pessimistic values of the functional, respectively;
δ^{op} and δ^{p} relative errors in percent.

In addition, the fulfillment of natural conditions is taken into account in the experiments:

$$\underline{c}_j \le \overline{c}_j, (j = \overline{1,N}), \underline{a}_{ij} \le \overline{a}_{ij}, \left(i = \overline{1,m}; j = \overline{1,N}\right), \underline{b}_i \le \overline{b}_i, (i = \overline{1,m}).$$

Here [z] denotes the integer part of the number z.

Table 1.

Results of the solved problems with two-digit coefficients ($N = 500; n = 300; m = 10$)

№	**1**	**2**	**3**	**4**	**5**
$\overline{f}_{op}$	22948.17	22737.03	22307.44	22490.98	21982.27
$\bar{f}_{op}^{d}$	23048.65	22993.03	22502.49	22615.05	22224.03
δ_{op}	0.436	1.113	0.867	0.54860	1.088
$\overline{f}_{pes}$	14039.38	14183.66	13947.47	13755.64	13584.62
$\bar{f}_{pes}^{d}$	14067.38	14310.70	14024.34	13781.12	13611.31
δ_{pes}	0.199	0.888	0.548	0.185	0.196

Table 2.

Results of the solved problems with two-digit coefficients $N = 1000;\ n = 600;\ m = 10$)

№	1	2	3	4	5
$\overline{f}_{op}$	45911.8	45296.4	44437.3	45092.6	44435.8
$\bar{f}_{op}^{d}$	45934.9	45362.8	44472.1	45158.1	44546.2
δ_{op}	0.050	0.147	0.078	0.145	0.248
$\overline{f}_{pes}$	27827.5	28181.9	28069.4	27822.5	27432.3
$\bar{f}_{pes}^{d}$	27849.6	28209.5	28134.9	27838.6	27451.2
δ_{pes}	0.080	0.097	0.233	0.058	0.069

Table 3.

Results of the solved problems with three-digit coefficients ($N = 500; n = 300; m = 10$)

№	1	2	3	4	5
$\overline{f}_{op}$	207813.4	204799.6	201112.6	203689.5	199601.7
$\bar{f}_{op}^{d}$	209053.5	206388.7	203216.6	204260.1	201176.2
δ_{op}	0.593	0.770	0.000	0.279	0.783
$\overline{f}_{pes}$	141571.2	142834.1	139843.9	138310.90	136465.8
$\bar{f}_{pes}^{d}$	141879.9	144340.4	140698.2	138635.1	136785.1
δ_{pes}	0.218	1.044	0.729	0.234	0.233

Table 4.

Results of the solved problems with three-digit coefficients ($N = 1000;\ n = 600;\ m = 10$)

№	1	2	3	4	5
$\overline{f}_{op}$	416772.4	407262.2	400559.0	410320.3	402729.9
$\bar{f}_{op}^{d}$	416857.3	408366.6	400862.6	410594.9	403404.3
δ_{op}	0.020	0.270	0.076	0.067	0.167
$\overline{f}_{pes}$	280754.5	284249.6	282822.2	280536.9	277027.7
$\bar{f}_{pes}^{d}$	280917.1	284610.0	283183.6	280718.5	277257.6
δ_{pes}	0.058	0.127	0.128	0.065	0.083

5. Conclusions

It can be seen from the above tables that the upper bounds obtained by the well-known linear programming method and by the method of this work do not differ significantly from each other. In this case, the relative errors of the obtained values vary within 0%-1.13%.

Hence, it can be seen that finding the upper bound for the optimistic and pessimistic values of the functional in problem (1)-(4) by the method of this work is more profitable than using the linear programming apparatus. Since, when using the linear programming method, the optimal values of the variables are obtained iteratively, and then the values of the functional. And in this work we directly obtain the upper bounds of the optimistic and pessimistic values of the function without determining the value of the arguments.

In addition, the tables show that with an increase in the number of variables, the relative errors decrease for both two-digit and three-digit coefficients. These facts once again confirm the effectiveness and naturalness of the developed method in this work.

Based on the above theoretical and computational materials, the following conclusions can be drawn:

The constructed majorizing function of the Lagrange-Type, in contrast to the classical function, contains only one type of variables – Lagrange multipliers.

The developed algorithm for minimizing this function is highly efficient, since the relative errors are rather smaller. And this once again shows the applicability of the developed algorithm for real applied problems.

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