# Representation of the solution of a system of fractional linear differential equations with Caputo derivatives for modeling processes in fractal media

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**ABSTRACT**

*An increasing amount of attention is currently being paid to the fundamentals of mathematical modeling of nonlocal heat and mass transfer processes in fractal media. Such physical processes are described by a system of fractional linear differential equations with Caputo derivatives. In this article we consider a linear Cauchy problem described by differential equations with fractional Caputo derivatives. Using an analogue of the Cauchy matrix we find an explicit representation of the solution to the problem in question.*

## 1. Introduction

The beginning of development of fractional integro-differential calculus dates back about three hundred years to the discussions of G. de L'Hôpital and G. Leibniz, and it is believed that the first step in fractional calculus was taken by L. Euler in 1738 [1]. He noticed that the results of calculating an integer derivative of a power function also made sense for a non-integer one. Then L. Laplace, J. Fourier, and others carried out studies in this area.

The study of linear ordinary differential equations with fractional operators largely began to develop more extensively with [1, 2].

Various optimal control problems for fractional dynamical systems are investigated in [3-5].

It is known that the representation of the solution of systems of linear differential equations corresponding to the systems of equations describing these processes plays an essential role in the investigation of various optimal control problems. Thus, the found representation of the solution of linear differential equations is used both in the problems of optimal control of linear systems and in the study of special cases in the problems of optimal control of nonlinear systems (see, e.g., [6, 7], etc.).

Various researchers have obtained representations of solutions of various differential equations with fractional derivatives by different methods. For instance, an initial problem for a fractional linear ordinary differential equation with Riemann-Liouville derivatives is investigated in [8]. Here the problem in question is reduced to an integral equation, and an explicit representation of the solution in terms of the Wright function is constructed. As a corollary of these results,

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necessary and sufficient conditions for the solvability of the Cauchy problem are obtained.

In [9] the Cauchy problem for linear systems of fractional ordinary differential equations with constant matrix coefficients is investigated and an analytical formula for its solution is obtained. It is shown that when the fractional order equals unity, the results obtained in that paper match the classical results.

In [10] for an ordinary differential equation with a fractional discretely distributed differentiation operator the initial problem is investigated, an explicit solution of the problem is found and the theorem of existence and uniqueness of the obtained solution is proved.

In [11] inhomogeneous linear systems of differential equations with classical Riemann-Liouville fractional derivatives as well as with regularized fractional Caputo derivatives are examined. Using the Laplace transform, the solutions of such systems are represented as analogues of the Cauchy formula for arbitrary measurable and bounded time functions in the right-hand side.

In the proposed study we investigate one linear inhomogeneous Cauchy problem for systems of fractional ordinary differential equations with Caputo derivatives. In contrast to previously known works, the representation of a solution is obtained using an analogue of the Cauchy matrix by the method presented in [12].

## 2. Some auxiliary facts

**Definition 1.** [1, 13-15] Suppose that $f(\cdot)$ is a locally integrable function on the interval $[a, b]$. When $t \in [a, b]$ and $\alpha > 0$, the left and right fractional Riemann-Liouville integrals are defined, respectively, by the relations

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \alpha \in R_+ $$

and

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} d\tau, \quad \alpha \in R_+, $$

where $\Gamma\ (\cdot)$ is Euler's gamma function (see, e.g. [1]).

**Definition 2.** Suppose that $f\ (\cdot)$ is an absolutely continuous function on the interval $[a, b]$. For $t\ \in\ [a, b]$ and $\alpha > 0$ the left and right fractional Riemann-Liouville integrals are defined, respectively, as

$$ {}_aD_t^\alpha f(t) = \frac{d^n}{dt^n}\left({}_aI_t^\alpha f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau, $$

and

$$ {}_tD_b^\alpha f(t) = \left(-\frac{d}{dt}\right)^n \left({}_tI_b^\alpha f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(\tau-t)^{1+\alpha-n}} d\tau, $$

where $n \in N$ is such that $n-1 < \alpha \leq n$, and $\Gamma\ (\cdot)$ is the same as in Definition 1.

***Definition 3.*** [13-15] Suppose that $f\ (\cdot) \in C^n[a, b]$. For $t \in [a, b]$ and $\alpha > 0$ the left and right fractional Caputo derivatives, respectively, are defined as

$$ {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau, $$

and

$$ {}_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(\tau)}{(\tau - t)^{1+\alpha-n}} d\tau, $$

where $n \in N$ is such that $n-1 < \alpha \leq n$.

In this article we deal with the left fractional Caputo derivative.

We say that $x(\cdot) \in AC\,\alpha([a,b],X)$, if there exists $\varphi(\cdot) \in L\infty([a,b],X)$ such that $x(t) = x(a) + ({}_aI_t^\alpha \varphi)(t), t \in [a,b]$.

**Theorem 1.** [1, 13-15] (Fractional integration by parts.) Suppose that $0 < \alpha < 1, f(\cdot)$ is a differentiable function on the interval $[a,b]$ and $g(\cdot) \in L1([a,b])$. Then the following formula for fractional integration by parts applies

$$ \int_a^b g(t) {}_a^C D_t^\alpha f(t) dt = \int_a^b f(t) {}_t D_b^\alpha g(t) dt + [{}_tI_b^\alpha g(t) f(t)]_a^b, $$

and

$$ \int_a^b g(t) {}_t^C D_b^\alpha f(t) dt = \int_a^b f(t) {}_a D_t^\alpha g(t) dt - [{}_aI_t^\alpha g(t) f(t)]_a^b. $$

## 3. Problem statement

Consider a linear equation system of the following type:

$$ {}_{t_0}^C D_t^\alpha x(t) = A(t)x(t) + f(t), \tag{1} $$

with the initial condition

$$ x(t_0) = x_0 \,. \tag{2} $$

Here $0 < \alpha < 1$, $A(t)$ is specified ($n$×$n$)-dimensional continuous matrix function, $t_0$ is specified start time, $x_0$ is specified initial state, $f(t)$ is specified $n$-dimensional continuous vector function, $x(t)$ is desired $n$-dimensional vector function, $x(t) \in C^n([t_0,t])$.

It is required to find the representation of the column vector $x(t)$ that satisfies Cauchy problem (1)-(2).

The function $x: [t_0,t_1] \to R^n$ is called a solution of the Cauchy problem (1), (2), if the inclusion $x(\cdot) \in AC\alpha\,(t_0,t_1], R^n)$ is satisfied, initial condition (2) is satisfied, and differential equation (1) is satisfied for almost every$t \in [t_0,t_1]$.

## 4. Representation of the solution

Suppose that $F(t,\tau)$ is an as yet unknown $(n \times n)$ – dimensional matrix function. By multiplying both sides of equation (1) by it, and integrating the resulting relation over $\tau$ from $t_0$ to $t$ , we obtain

$$ \int_{t_0}^t F(t,\tau) {}_{t_0}^C D_\tau^\alpha x(\tau)\, d\tau = \int_{t_0}^t F(t,\tau) A(\tau) x(\tau)\, d\tau + \int_{t_0}^t F(t,\tau) f(\tau)\, d\tau. \tag{3} $$

Using the formula of partial integration (see, e.g., [15]), we have

$$\int_{t_0}^{t} F(t,\tau)\, {}_{t_0}^{C}D_{\tau}^{\alpha} x(\tau)d\tau = \int_{t_0}^{t} {}_{\tau}D_{t_1}^{\alpha} F(t,\tau)x(\tau)d\tau + {}_{\tau}I_{t_1}^{1-\alpha}F(t,\tau)x(\tau)\big|_{t_0}^{t} =$$

$$= \int_{t_0}^{t} {}_{\tau}D_{t_1}^{\alpha} F(t,\tau)x(\tau)d\tau + {}_{t}I_{t_1}^{1-\alpha}F(t,t)x(t) - {}_{t}I_{t_1}^{1-\alpha}F(t,t_0)x(t_0).$$

Then formula (3) can be written as follows:

$$\int_{t_0}^{t} {}_{t_0}D_{\tau}^{\alpha} F(t,\tau)x(\tau)d\tau + {}_{t}I_{t_1}^{1-\alpha}F(t,t)x(t) - {}_{t}I_{t_1}^{1-\alpha}F(t,t_0)x(t_0) =$$

$$= \int_{t_0}^{t} F(t,\tau)A(\tau)x(\tau)\, d\tau + \int_{t_0}^{t} F(t,\tau)f(\tau)\, d\tau. \tag{4}$$

Suppose the matrix function $F(t,\tau)$ is the solution of the matrix equation

$$ {}_{t_0}D_{\tau}^{\alpha} F(t,\tau) = F(t,\tau)A(\tau), \tag{5}$$

$$ {}_{t}I_{t_1}^{1-\alpha}F(t,t) = E, \tag{6}$$

where $E$ is a unit matrix. It is assumed that the matrix function satisfies those smoothness conditions that are necessary for further discussion.

Hence, we arrive at the relation

$$x(t) = {}_{t}I_{t_1}^{1-\alpha}F(t,t_0)x(t_0) + \int_{t_0}^{t} F(t,\tau)f(\tau)\, d\tau. \tag{7}$$

Thus, the following statement is proved.

**Theorem 2.** The solution $x(t)$ of a system of fractional linear ordinary differential equations with Caputo derivatives (1) allows a representation in the form (7).

Here $F(t,\tau)$ is the solution of linear matrix equations (5) and (6) and is the fundamental matrix for problem (1), (2).

As noted in the introduction, the obtained result will be used in the investigation of optimal control problems described by systems of fractional differential equations.

## 5. Conclusion

The article investigates one linear Cauchy problem for systems of fractional ordinary differential equations with left-hand fractional Caputo derivatives. Using the Cauchy matrix, a representation of the solution of the problem in the explicit integral form is obtained.

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