A necessary optimality condition of the Pontryagin maximum principle type in one problem of optimal control of a system with distributed parameters

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ABSTRACT

We consider a variable-structure optimal control problem described in different domains by a hyperbolic integro-differential equation and a Volterra integral equation, respectively. The quality functional is terminal. A formula for the increment of the quality criterion is constructed and an analogue of L.S. Pontryagin's maximum principle is proved by investigating on special McShane-type variations.

1. Introduction

In [1-4] etc., a number of multistage optimal control problems described in different time intervals by different equations are investigated.

Such optimal control problems are also called variable-structure optimal control problems (see e.g. [1-6]).

In this study, one problem of optimal control of a variable-structure object described in different domains by hyperbolic integro-differential equations and Volterra integral equations is considered.

An analogue of L.S. Pontryagin's maximum principle is established.

2. Problem statement

Suppose that $D_i = [t_{i-1}, t_i] \times [x_0, x_1], i = 1,2$ are specified rectangles, with $t_0 < t_1 < t_2\, ; x_0 < x_1 < x_2$.

Suppose that the controlled process in the domain $D = D_1 \cup D_2$ is described by the boundary problem

$$\frac{\partial^2 z_1(t,x)}{\partial t \partial x} = \int_{t_0}^{t}\int_{x_0}^{x} f_1\big(t,x,\tau,s,z_1(\tau,s),u_1(\tau,s)\big)ds d\tau\,, (t,x) \in D_1, \tag{1}$$

$$z_1(t_0,x) = a(x), x \in [x_0,x_1],$$
$$z_1(t,x_0) = b(t), t \in [t_0,t_1] \tag{2}$$

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and the system of two-dimensional Volterra integral equations

$$z_2(t,x)=\int_{t_1}^{t}\int_{x_0}^{x} f_2\big(t,x,\tau,s,z_2(\tau,s),u_2(\tau,s)\big)dsd\tau+G\big(z_1(t_1,x)\big),(t,x)\in D_2. \quad (3)$$

Here, $f_i(t,x,\tau,s,z_i,u_i), i=1,2$ are specified n-dimensional vector functions continuous in the set of variables with partial derivatives in $z_i, i=1,2$, $a(x)$ and $b(t)$ are specified n-dimensional absolutely continuous vector functions, $G(z_1)$ is specified n-dimensional continuously differentiable vector function, $U_1\subset R^r, U_2\subset R^q$ are specified non-empty and bounded sets, $u_1(t,x)\big(u_2(t,x)\big)$ is a $r(q)$-dimensional is a measurable and bounded vector function of control actions, satisfying the constraint

$$u_1(t,x)\in U_1\subset R^r,(t,x)\in D_1. \quad (4)$$

$$u_2(t,x)\in U_2\subset R^q,(t,x)\in D_2. \quad (5)$$

We shall call the pair $\big(u_1(t,x),u_2(t,x)\big)$ with the above properties an admissible control.

Suppose that a unique, absolutely continuous solution $z_1(t,x)$ of boundary value problem (1)-(2) and a unique continuous solution $z_2(t,x)$ of integral equation (3) correspond to each given admissible control $\big(u_1(t,x),u_2(t,x)\big)$.

On the solutions of problem (1)-(3) given by all possible admissible controls $\big(u_1(t,x),u_2(t,x)\big)$, we shall determine the terminal functional

$$J(u_1,u_2)=\varphi_1\big(z_1(t_1,x)\big)+\varphi_2\big(z_2(t_2,x)\big). \quad (6)$$

where $\varphi_i(z_i), i=1,2$ are specified continuously differentiable scalar functions.

The problem is to find the minimum value of terminal functional (6) under constraints (1)-(5).

The admissible control $\big(u_1(t,x),u_2(t,x)\big)$ that satisfies a minimum value to functional (6) shall be called the optimal control, and the corresponding process $\big(u_1(t,x),u_2(t,x),z_1(t,x),z_2(t,x)\big)$ shall be called the optimal process.

The aim of the study is to derive a necessary optimality condition of the Pontryagin maximum principle type in the problem under consideration.

3. The formula for the increment of the quality functional

Suppose that $\big(u_1(t,x),u_2(t,x),z_1(t,x),z_2(t,x)\big)$ and $\big(\bar{u}_1(t,x)=u_1(t,x)+\Delta u_1(t,x)$, $\bar{u}_2(t,x)=u_2(t,x)+\Delta u_2(t,x)$, $\bar{z}_1(t,x)=z_1(t,x)+\Delta z_1(t,x)$, $\bar{z}_2(t,x)=z_2(t,x)+\Delta z_2(t,x)\big)$ are some admissible processes. It is clear then that $\big(\Delta z_1(t,x),\Delta z_2(t,x)\big)$ will be the solution of the problem

$$\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}=\int_{t_0}^{t}\int_{x_0}^{x}\big[f_1\big(t,x,\tau,s,\bar{z}_1(\tau,s),\bar{u}_1(\tau,s)\big)-f_1\big(t,x,\tau,s,z_1(\tau,s),u_1(\tau,s)\big)\big]dsd\tau, \quad (7)$$

$$\Delta z_1(t_0,x)=0,$$
$$\Delta z_1(t,x_0)=0. \quad (8)$$

$$\Delta z_2(t,x)=\int_{t_1}^{t}\int_{x_0}^{x}\big[f_2\big(t,x,\tau,s,\bar{z}_2(\tau,s),\bar{u}_2(\tau,s)\big)-f_2\big(t,x,\tau,s,z_2(\tau,s),u_2(\tau,s)\big)\big]dsd\tau+$$
$$+G\big(\bar{z}_1(t_1,x)\big)-G\big(z_1(t_1,x)\big), \quad (9)$$

and the increment of functional (6) has the form

$$J(\bar{u}_1,\bar{u}_2)-J(u_1,u_2)=\big[\varphi_1\big(\bar{z}_1(t_1,x)\big)-\varphi_1\big(z_1(t_1,x)\big)\big]+\big[\varphi_2\big(\bar{z}_2(t_2,x)\big)-\varphi_2\big(z_2(t_2,x)\big)\big]. \quad (10)$$

Suppose that $\psi_i(t,x), i=1,2$ are as yet unknown vector functions.

From identities (7) and (9), we have that

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1}\psi_1{}'(t,x)\frac{\partial^2\Delta z_1(t,x)}{\partial t\partial x}dxdt=\int_{t_0}^{t_1}\int_{x_0}^{x_1}\psi_1{}'(t,x)\left[\int_{t_0}^{t}\int_{x_0}^{x}\left[f_1\big(t,x,\tau,s,\bar z(\tau,s),\bar u(\tau,s)\big)-\right.\right.$$
$$\left.\left.-f_1\big(t,x,\tau,s,z(\tau,s),u(\tau,s)\big)\right]dsd\tau\right]dxdt, \tag{11}$$

$$\int_{t_1}^{t_2}\int_{x_0}^{x_1}\psi_2{}'(t,x)\,\Delta z_2(t,x)dxdt=$$
$$=\int_{t_1}^{t_2}\int_{x_0}^{x_1}\psi_2{}'(t,x)\left[\int_{t_1}^{t}\int_{x_0}^{x}\left[f_2\big(t,x,\tau,s,\bar z_2(\tau,s),\bar u_2(\tau,s)\big)-\right.\right.$$
$$\left.\left.-f_2\big(t,x,\tau,s,z_2(\tau,s),u_2(\tau,s)\big)\right]dsd\tau dxdt+\right.$$
$$+\int_{t_1}^{t_2}\int_{x_0}^{x_1}\psi_2{}'(t,x)\,[G(\bar z_1(t_1,x))-G(z_1(t_1,x))]dxdt, \tag{12}$$

From formula (11), given boundary conditions (8) and applying the Fubini formula (see e.g., [7]), we have that

$$\psi_1'(t_1,x_1)\Delta z_1(t_1,x_1)-\int_{t_0}^{t_1}\frac{\partial\psi_1'(t,x)}{\partial t}\Delta z_1(t_1,x)dt-\int_{x_0}^{x_1}\frac{\partial\psi_1'(t,x)}{\partial x}\Delta z_1(t,x_1)dx+$$
$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\frac{\partial^2\psi_1'(t,x)}{\partial t\partial x}\Delta z_1(t,x)dxdt=$$
$$=\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{t}^{t_1}\int_{x}^{x_1}\psi_1{}'(\tau,s)\left[f_1\big(\tau,s,t,x,\bar z_1(t,x),\bar u_1(t,x)\big)-\right.\right.$$
$$\left.\left.-f_1\big(\tau,s,t,x,z_1(t,x),u_1(t,x)\big)\right]dsd\tau\right]dxdt. \tag{13}$$

And from identity (12) by introducing the notation

$$M(\psi_2(t,x),z_1)=\psi_2{}'(t,x)G(z_1)$$

and applying the Fubini formula, we have that

$$\int_{t_1}^{t_2}\int_{x_0}^{x_1}\psi_2{}'(t,x)\,\Delta z_2(t,x)dxdt=$$
$$=\int_{t_1}^{t_2}\int_{x_0}^{x_1}\left[\int_{t}^{t_2}\int_{x}^{x_1}\psi_2{}'(t,x)\left[f_2\big(\tau,s,t,x,\bar z_2(t,x),\bar u_2(t,x)\big)-\right.\right.$$
$$\left.\left.-f_2\big(\tau,s,t,x,z_2(t,x),u_2(t,x)\big)\right]dsd\tau\right]dxdt+$$
$$+\int_{t_1}^{t_2}\int_{x_0}^{x_1}\left[M\big(\psi_2(t,x),\bar z_1(t_1,x)\big)-M\big(\psi_2(t,x),z_1(t_1,x)\big)\right]dxdt, \tag{14}$$

From the formula for increment (10) of the quality functional, using the Taylor formula, we obtain

$$J(\bar u_1,\bar u_2)-J(u_1,u_2)=\frac{\partial\varphi_1'\big(z_1(t_1,x_1)\big)}{\partial z_1}\Delta z_1(t_1,x_1)+\frac{\partial\varphi_2'\big(z_2(t_1,x_1)\big)}{\partial z_2}\Delta z_2(t_2,x_1)+$$
$$+o_1(\|\Delta z_1(t_1,x_1)\|)+o_2(\|\Delta z_2(t_1,x_1)\|). \tag{15}$$

Here and further $\|\alpha\|$ means the norm of the vector $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)'$ determined by the formula $\|\alpha\|=\sum_{i=1}^n|\alpha_i|$, $o(\alpha)$ is a quantity of a higher order than α, i.e., $\frac{o(\alpha)}{\alpha}\to 0$ at $\alpha\to 0$, and the dash ($'$) is a scalar product operation for vectors, and a transpose operation for matrices.

Given identity (9) we can write that

$$\frac{\partial \varphi_2'\big(z_2(t_1,x_1)\big)}{\partial z_2}\Delta z_2(t_2,x_1)=\frac{\partial \varphi_2'\big(z_2(t_1,x_1)\big)}{\partial z_2}\int_{t_1}^{t_2}\int_{x_0}^{x_1}\big[f_2\big(t_2,x_1,t,x,\bar z_2(t,x),\bar u_2(t,x)\big)-$$

$$-f_2\big(t_2,x_1,t,x,z_2(t,x),u_2(t,x)\big)\big]dxdt+\frac{\partial \varphi_2'\big(z_2(t_1,x_1)\big)}{\partial z_2}[G(\bar z_1(t_1,x_1))-G(z_1(t_1,x_1))].$$

We introduce the notation

$$N(z_1)=\frac{\partial \varphi_2'\big(z_2(t_1,x_1)\big)}{\partial z_2}G(z_1),$$

$$H_1\big(x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)=\int_{t}^{t_1}\int_{x}^{x_1}\psi_1{}'(\tau,s)f_1\big(\tau,s,t,x,z_1(t,x),u_1(t,x)\big)dsd\tau,$$

$$H_2\big(x,z_2(t,x),u_2(t,x),\psi_2(t,x)\big)=-\frac{\partial \varphi_2'\big(z_2(t_1,x_1)\big)}{\partial z_2}f_2\big(t_2,x_1,t,x,z_2(t,x),u_2(t,x)\big)+$$

$$+\int_{t}^{t_2}\int_{x}^{x_1}\psi_2{}'(\tau,s)f_2\big(\tau,s,t,x,z_2(t,x),u_2(t,x)\big)dsd\tau.$$

Using the Taylor formula, we obtain that

$$N\big(\bar z_1(t_1,x_1)\big)-N\big(z_1(t_1,x_1)\big)=\frac{\partial N'\big(z_1(t_1,x_1)\big)}{\partial z_1}\Delta z_1(t_1,x_1)+o_3(\|z_1(t_1,x_1)\|), \qquad (16)$$

$$M\big(\psi_2(t,x),\bar z_1(t_1,x)\big)-M\big(\psi_2(t,x),z_1(t_1,x)\big)=\frac{\partial M'\big(\psi_2(t,x),z_1(t_1,x)\big)}{\partial z_1}\Delta z_1(t_1,x_1)+$$

$$+o_4(t\,;\|z_1(t_1,x)\|). \qquad (17)$$

Taking into account the introduced notations and formulas (13), (14), (16), (17), increment (15) of the functional is represented in the form

$$J(\bar u_1,\bar u_2)-J(u_1,u_2)=\frac{\partial \varphi_1'\big(z_1(t_1,x_1)\big)}{\partial z_1}\Delta z_1(t_1,x_1)+\psi_1'(t_1,x_1)\,\Delta z_1(t_1,x_1)-$$

$$\int_{t_0}^{t_1}\frac{\partial \psi_1'(t,x)}{\partial t}\Delta z_1(t_1,x)dt-\int_{x_0}^{x_1}\frac{\partial \psi_1'(t,x_1)}{\partial x}\Delta z_1(t,x_1)dx+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\frac{\partial^2 \psi_1'(t,x)}{\partial t\partial x}\Delta z_1(t,x)dxdt+$$

$$+\frac{\partial N'\big(z_1(t_1,x_1)\big)}{\partial z_1}\Delta z_1(t_1,x_1)-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\frac{\partial M'\big(\psi_2(t,x),z_1(t_1,x)\big)}{\partial z_1}\Delta z_1(t,x)\,dxdt+$$

$$+\int_{t_1}^{t_2}\int_{x_0}^{x_1}\psi_2{}'(t,x)\Delta z_2(t,x)\,dxdt-$$

$$-\int_{t_0}^{t}\int_{x_0}^{x}[H_1(t,x,\bar z(t,x),\bar u(t,x),\psi_1(t,x))-H_1(t,x,z(t,x),u(t,x),\psi_1(t,x))]dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\big[H_2\big(t,x,\bar z_2(t,x),\bar u_2(t,x),\psi_2(t,x)\big)-H_2\big(t,x,z_2(t,x),u_2(t,x),\psi_2(t,x)\big)\big]dxdt+$$

$$+o_1(\|\Delta z_1(t_1,x_1)\|)+o_2(\|\Delta z_2(t_1,x_1)\|)+o_3(\|z_1(t_1,x_1)\|)-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}o_4(t\,;\|z_1(t_1,x)\|)\,dxdt. \qquad (18)$$

Taking into account the smoothness conditions imposed on the Hamilton-Pontryagin functions $H_i\big(t,x,z_i(t,x),u_i(t,x),\psi_i(t,x)\big), i=1{,}2$ in $z_i, i=1{,}2$, after some manipulations we obtain that

$$H_i\big(t,x,\bar z_i(t,x),\bar u_i(t,x),\psi_i(t,x)\big)-H_i\big(t,x,z_i(t,x),u_i(t,x),\psi_i(t,x)\big)=$$

$$=H_i\big(t,x,z_i(t,x),\bar u_i(t,x),\psi_i(t,x)\big)-H_i\big(t,x,z_i(t,x),\bar u_i(t,x)),\psi_i(t,x)\big)+$$

$$+\frac{\partial H_i{}'\big(t,x,z_i(t,x),u_i(t,x),\psi_i(t,x)\big)}{\partial z_i}\Delta z_i(t,x)+\left[\frac{\partial H_i\big(t,x,z_i(t,x),\bar{u}_i(t,x),\psi_i(t,x)\big)}{\partial z_i}-\right.$$

$$\left.-\frac{\partial H_i\big(t,x,z_i(t,x),u_i(t,x),\psi_i(t,x)\big)}{\partial z_i}\right]'\Delta z_i(t,x)+o_{i+4}(\|\Delta z_i(t,x)\|),i=1,2.$$

Given this formula in increment formula (18), we will have

$$J(\bar{u}_1,\bar{u}_2)-J(u_1,u_2)=\frac{\partial\varphi_1'\big(z_1(t_1,x_1)\big)}{\partial z_1}\Delta z_1(t_1,x_1)+\psi_1'^{(t_1,x_1)}\Delta z_1(t_1,x_1)-$$

$$-\int_{t_0}^{t_1}\frac{\partial\psi_1'(t_1,x)}{\partial t}\Delta z_1(t_1,x)dt-\int_{x_0}^{x_1}\frac{\partial\psi_1'(t,x_1)}{\partial x}\Delta z_1(t,x_1)dx+$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\frac{\partial^2\psi_1'(t,x)}{\partial t\partial x}\Delta z_1(t,x)dxdt+\frac{\partial N'\big(z_1(t_1,x_1)\big)}{\partial z_1}\Delta z_1(t_1,x_1)-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\frac{\partial M'\big(\psi_2(t,x),z_1(t_1,x)\big)}{\partial z_1}\Delta z_1(t_1,x)\,dxdt+\int_{t_1}^{t_2}\int_{x_0}^{x_1}\psi_2{}'(t,x)\Delta z_2(t,x)\,dxdt-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}[H_1(t,x,\bar{z}_1(t,x),\bar{u}_1(t,x),\psi_1(t,x))-H_1(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x))]dxdt-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\frac{\partial H_1{}'\big(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)}{\partial z_1}\Delta z_1(t,x)\,dxdt-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\frac{\partial H_1\big(t,x,z_i(t,x),\bar{u}_1(t,x),\psi_1(t,x)\big)}{\partial z_1}-\right.$$

$$\left.-\frac{\partial H_1\big(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)}{\partial z_1}\right]'\Delta z_1(t,x)dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\big[H_2\big(t,x,\bar{z}_2(t,x),\bar{u}_2(t,x),\psi_2(t,x)\big)-H_2\big(t,x,z_2(t,x),u_2(t,x),\psi_2(t,x)\big)\big]dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\frac{\partial H_2{}'\big(t,x,z_2(t,x),u_2(t,x),\psi_2(t,x)\big)}{\partial z_2}\Delta z_2(t,x)\,dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}\left[\frac{\partial H_2\big(t,x,z_2(t,x),\bar{u}_2(t,x),\psi_2(t,x)\big)}{\partial z_2}-\right.$$

$$\left.-\frac{\partial H_2\big(t,x,z_2(t,x),u_2(t,x),\psi_2(t,x)\big)}{\partial z_2}\right]'\Delta z_2(t,x)dxdt-$$

$$+o_1(\|\Delta z_1(t_1,x_1)\|)+o_2(\|\Delta z_2(t_1,x_1)\|)+o_3(\|z_1(t_1,x_1)\|)-\int_{t_1}^{t_2}\int_{x_0}^{x_1}o_4(t;\|z_1(t_1,x)\|)\,dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1}o_5(\|z_1(t,x)\|)\,dxdt-\int_{t_1}^{t_2}\int_{x_0}^{x_1}o_6(\|z_1(t,x)\|)\,dxdt. \quad (19)$$

Suppose that the vector function $\psi_i(t,x), i=\overline{0,p}$ satisfy the relations

$$\frac{\partial^2\psi_i(t,x)}{\partial t\partial x}=\frac{\partial H_1\big(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)}{\partial z_1}, \quad (20)$$

$$\frac{\partial\psi_1(t,x_1)}{\partial t}=0,$$

$$\frac{\partial\psi_1(t_1,x)}{\partial x} = -\frac{\partial M\big(\psi_2(t,x), z_1(t_1,x)\big)}{\partial z_1}, \tag{21}$$

$$\psi_1(t_1,x_1) = -\frac{\partial\varphi_1\big(z_1(t_1,x_1)\big)}{\partial z_1} - \frac{\partial N\big(z_1(t_1,x_1)\big)}{\partial z_1}, \tag{22}$$

$$\psi_2(t,x) = \frac{\partial H_2\big(t,x,z_2(t,x),u_2(t,x),\psi_2(t,x)\big)}{\partial z_2}. \tag{23}$$

Relations (20)-(21) are a linear boundary value problem for $\psi_1(t,x)$, and relation (23) is a linear integral equation with respect to $\psi_2(t,x)$.

Following the classical terminology, we shall call them conjugate systems.

If relations (20)-(23) are fulfilled, increment formula (19) will take the form

$$J(\bar u_1,\bar u_2) - J(u_1,u_2) = -\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\int_{x_0}^{x_1}\big[H_i\big(t,x,z_i(t,x),\bar u_i(t,x),\psi_i(t,x)\big) -$$
$$-H_i\big(t,x,z_i(t,x),u_i(t,x),\psi_i(t,x)\big)\big]dxdt -$$
$$-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\int_{x_0}^{x_1}\left[\frac{\partial H_i\big(t,x,z_i(t,x),\bar u_i(t,x),\psi_i(t,x)\big)}{\partial z_i} -\right.$$
$$\left.-\frac{\partial H_1\big(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)}{\partial z_1}\right]' \Delta z_i(t,x)dxdt -$$
$$+\sum_{i=1}^{2} o_i(\|\Delta z_i(t_i,x_1)\|) + o_3(\|z_1(t_1,x_1)\|) - \int_{t_1}^{t_2}\int_{x_0}^{x_1} o_4(t\,;\|z_1(t,x)\|)\,dxdt -$$
$$-\sum_{i=1}^{2}\int_{t_1}^{t_2}\int_{x_0}^{x_1} o_{i+4}(\|z_i(t,x)\|)\,dxdt. \tag{24}$$

Suppose that $\Delta u_2(t,x)=0$. Then from relations (7)-(10) we obtain that

$$\Delta z_1(t,x) = \int_{t_0}^{t}\int_{x_0}^{x}\big[f_1\big(t,x,\tau,s,\bar z_1(\tau,s),\bar u_1(\tau,s)\big) - f_1\big(t,x,\tau,s,z_1(\tau,s),u_1(\tau,s)\big)\big]dsd\tau, \tag{25}$$

$$\Delta z_2(t,x) = \int_{t_2}^{t}\int_{x_0}^{x}\big[f_2\big(t,x,\tau,s,\bar z_2(\tau,s),\bar u_2(\tau,s)\big) - f_2\big(t,x,\tau,s,z_2(\tau,s),u_2(\tau,s)\big)\big]dsd\tau +$$
$$+G\big(\bar z_1(t_1,x_1)\big) - G\big(z_1(t_1,x_1)\big). \tag{26}$$

Assuming that $\Delta u_1(t,x)=0$, then from (7)-(9) and (10) we obtain that

$$\Delta z_1(t,x) = 0,$$

$$\Delta z_2(t,x) = \int_{t_2}^{t}\int_{x_0}^{x}\big[f_2\big(t,x,\tau,s,\bar z_2(\tau,s),\bar u_2(\tau,s)\big) - f_2\big(t,x,\tau,s,z_2(\tau,s),u_2(\tau,s)\big)\big]dsd\tau. \tag{27}$$

From formula (25), applying the Gronwall-Wendorff lemma (see, e.g., [8]), after some manipulations we obtain the estimate

$$\|\Delta z_1(t,x)\| \le L_1\int_{t_0}^{t}\int_{x_0}^{x}\left[\int_{\tau}^{t}\int_{s}^{x}\big\|f_1\big(t,x,\tau,s,z(\tau,s),\bar u(\tau,s)\big) -\right.$$
$$\left.-f_1\big(t,x,\tau,s,z(\tau,s),u(\tau,s)\big)\big\|dxdt\right]dsd\tau, \tag{28}$$

where $L_1 = const > 0$ is a constant.

From relation (26), applying the Gronwall-Wendorff lemma, we arrive at the estimate

$$\|\Delta z_2(t,x)\| \le L_2\|\Delta z_1(t,x)\|, (t,x)\in D_2 \tag{29}$$

($L_2 = const > 0$ is a constant).

And from formula (27) we similarly obtain the estimate

$$\|\Delta z_2(t,x)\| \le L_3 \int_{t_1}^{t}\int_{x_0}^{x} \left\| f_1\big(t,x,\tau,s,z(\tau,s),\bar{u}(\tau,s)\big) - f\big(t,x,\tau,s,z(\tau,s),u(\tau,s)\big)\right\| ds d\tau. \quad (30)$$

($L_3 = const > 0$ is a constant).

4. Necessary optimality condition

Suppose that $\Delta u_2(t,x) = 0$. Then it follows from increment formula (24) that

$$J(\bar{u}_1,u_2) - J(u_1,u_2) = -\int_{t_0}^{t_1}\int_{x_0}^{x_1}\big[H_1\big(t,x,z_1(t,x),\bar{u}_1(t,x),\psi_1(t,x)\big) -$$
$$-H_1\big(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)\big]dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\frac{\partial H_1{}'\big(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)}{\partial z_1}\Delta z_1(t,x)\,dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\frac{\partial H_1\big(t,x,z_i(t,x),\bar{u}_1(t,x),\psi_1(t,x)\big)}{\partial z_1} -\right.$$
$$\left.-\frac{\partial H_1\big(t,x,z_1(t,x),u_1(t,x),\psi_1(t,x)\big)}{\partial z_1}\right]'\Delta z_1(t,x)dxdt +$$
$$+o_1(\|\Delta z_1(t_1,x_1)\|) + o_2(\|\Delta z_2(t_1,x_1)\|) + o_3(\|z_1(t_1,x_1)\|) - \int_{t_1}^{t_2}\int_{x_0}^{x_1} o_4(t\,;\|z_1(t,x)\|)\,dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} o_5(\|z_1(t,x)\|)\,dxdt - \int_{t_1}^{t_2}\int_{x_0}^{x_1} o_6(\|z_1(t,x)\|)\,dxdt. \quad (31)$$

Suppose that $(\theta,\xi) \in [t_0,t_1) \times [x_0,x_1)$ is an arbitrary Lebesgue point of the control $u_1(t,x)$, $v_1 \in U_1$ is an arbitrary vector, and $\varepsilon > 0$ is an arbitrary sufficiently small number such that $\theta + \varepsilon < t_1$ and $\xi + \varepsilon < x_1$.

The special increment of the control function $u_1(t,x)$ will be determined from the formula

$$\Delta u_1(t,x\,;\varepsilon) = \begin{cases} v_1 - u_1(t,x), (t,x) \in [\theta,\theta+\varepsilon) \times [\xi,\xi+\varepsilon), \\ 0, (t,x) \in D_1 \backslash [\theta,\theta+\varepsilon) \times [\xi,\xi+\varepsilon). \end{cases} \quad (32)$$

Given this formula and estimates (26), (29) from increment formula (31) we get the validity of the expansion

$$J\big(u_1(t,x) + \Delta u_1(t,x\,;\varepsilon), u_2(t,x)\big) - J\big(u_1(t,x),u_2(t,x)\big) =$$
$$= -\varepsilon^2\big[H_1\big(\theta,\xi,z_1(\theta,\xi),v_1,\psi_1(\theta,\xi)\big) - H_1\big(\theta,\xi,z_1(\theta,\xi),u_1(\theta,\xi),\psi_1(\theta,\xi)\big)\big] + o(\varepsilon^2). \quad (33)$$

Suppose now that $\Delta u_1(t,x) = 0$, and the special increment of the control function $u_2(t,x)$ will be determined from the formula

$$\Delta u_2(t,x\,;\varepsilon) = \begin{cases} v_2 - u_2(t,x), (t,x) \in [\theta,\theta+\mu) \times [\xi,\xi+\mu), \\ 0, (t,x) \in D_2 \backslash [\theta,\theta+\mu) \times [\xi,\xi+\mu). \end{cases} \quad (34)$$

Here, $v_2 \in U_2$ is an arbitrary vector, $(\theta,\xi) \in [t_0,t_1) \times [x_0,x_1)$ is an arbitrary Lebesgue point of the control $u_2(t,x)$, and $\mu > 0$ is an arbitrary sufficiently small number such that $\theta + \mu < t_2$ and $\xi + \mu < x_1$.

Given estimate (30) and formula (34) from increment formula (24) of the quality functional, we will have

$$J\big(u_1(t,x),u_2(t,x) + \Delta u_2(t,x\,;\mu)\big) - J\big(u_1(t,x),u_2(t,x)\big) =$$

$$= -\mu^2\left[H_1\left(\theta,\xi,z_2(\theta,\xi),v_2,\psi_2(\theta,\xi)\right) - H_2\left(\theta,\xi,z_2(\theta,\xi),u_2(\theta,\xi),\psi_2(\theta,\xi)\right)\right] + o(\mu^2). \quad (35)$$

From expansions (33) and (35) it follows that the following statement is true.

Theorem. The optimality of the admissible control $\left(u_1(t,x),u_2(t,x)\right)$ requires that the condition of the maximum

$$\max_{v_1\in U_1} H_1(\theta,\xi,z_1(\theta,\xi),v_1,\psi_1(\theta,\xi)) = H_1(\theta,\xi,z_1(\theta,\xi),u_1(\theta,\xi),\psi_1(\theta,\xi)),$$

$$\max_{v_2\in U_2} H_2(\theta,\xi,z_2(\theta,\xi),v_2,\psi_2(\theta,\xi)) = H_2(\theta,\xi,z_2(\theta,\xi),u_2(\theta,\xi),\psi_2(\theta,\xi))$$

should be true for all $(\theta,\xi) \in [t_0,t_1) \times [x_0,x_1)$ and $(\theta,\xi) \in [t_1,t_2) \times [x_0,x_1)$, respectively.

The proved theorem is an analogue of the Pontryagin maximum principle for the problem under investigation.

5. Conclusion

This study considers one problem of optimal control of a variable structure with distributed parameters described by a set of hyperbolic integro-differential equations and Volterra integral equations.

Applying one variant of the method of increments, an analogue of the Pontryagin maximum principle is proved.

References

[1] В.Н. Розова, Оптимальное управление ступенчатыми системами с неинтегральным функционалом, Вестник РУДН, Сер. прикл. матем. и компьютерная математика. 1 No.1 (2002) pp.131-136. [In Russian: V. N. Rozova, Optimal control of step systems with non-integral functional, Vestnik RUDN, Ser. prikladnoi matem. i komputernaya matematika].

[2] Р.Р. Исмайлов, К.Б. Мансимов, Об условиях оптимальности в одной ступенчатой задаче управления, Журн. Выч. мат. и мат. физики. No.10 (2006) pp.1758-1770. [In Russian: R.R. Ismaylov, K.B. Mansimov, On optimality conditions in one step control problem, Zhurn. Vych. mat. i mat. fiziki.].

[3] Г.К. Захаров, Оптимизация ступенчатых систем управления, Автоматика и телемеханика. No.8 (1981) pp.3-9. [In Russian: G.K. Zakharov, Optimization of step control systems, Avtomatika i telemehanika].

[4] М.С. Никольский, Об одной вариационной задаче с переменной структурой, Вестник МГУ, Серия Выч. мат. и кибернетика. No.2 (1987) pp.36-41. [In Russian: M.S. Nikolsky, On one variational variable-structure problem, Vestnik MGU, Seria Vych. mat. i kibernetika].

[5] К.Б. Мансимов, Ш. Ш. Сулейманова, К оптимальности особых в классическом смысле управлений в одной задаче оптимального управления системами с переменной структурой, Вестник Томского гос. Университета. No.44 (2018) pp.10-24. [In Russian: K.B. Mansimov, Sh.Sh. Suleymanova, On optimality of special in the classical sense controls in one problem of optimal control of variable-structure systems, Vestnik Tomskogo gos. universiteta].

[6] А.Н. Кириллов, Динамические системы с переменной структурой и размерностью, Изв. Вузов, Приборостроение. 52 No.3 (2009) pp.23-28. [In Russian: A.N. Kirillov, Dynamic variable-structure and dimensionality systems, Izv. Vuzov, Priborostroenie].

[7] А.Н. Колмогоров, С.В. Фомин, Элементы теории функций и функционального анализа, М. Наука, (2004) 286 p. [In Russian: A.N. Kolmogorov, S.V. Fomin, Elements of Function Theory and Functional Analysis, Moscow, Nauka].

[8] М.М. Новоженов, В.И. Сумин, М.И. Сумин, Методы оптимального управления системами математической физики, Горький, Изд-во ГГУ, (1986) 87 p. [In Russian: M.M. Novozhenov, V.I. Sumin, M.I. Sumin, Methods of optimal control of mathematical physics systems, Gorky, Izd-vo GGU.].