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An analogue of Euler's equation and second-order necessary optimality conditions in one N.N. Moiseyev-type optimal control problem

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ABSTRACT

We consider an optimal control problem described by a system of ordinary differential equations with a non-type, multipoint quality functional. An analogue of Euler's equation is proved under the assumption of openness of the control domain and a number of second-order necessary optimality conditions are established.

1. Introduction

In his monograph [1], N.N. Moiseyev considered one optimal control problem described by a system of ordinary differential equations with a non-type quality criterion and proved the necessary optimality condition in the form of L.S. Pontryagin's maximum principle [2-4].

In this study we consider a similar optimal control problem, but with a more general, multipoint quality functional.

Under the assumption of openness of the control domain, we prove an analogue of Euler's equation and establish the second-order necessary optimality conditions.

2. Problem statement

Suppose that the controlled continuous process on a given time segment $[t_0, t_1]$ $(t_0 < t_1)$ is described by a system of ordinary differential equations

$$\dot{x} = f(t, x, u), \qquad t \in [t_0, t_1] \tag{1}$$

with the initial condition

$$x(t_0) = x_0. \tag{2}$$

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Here, $f(t,x,u)$ is a specified n-dimensional vector function that is continuous in the set of variables together with partial derivatives in (x,u) up to and including the second order, x_0 is a specified constant initial vector, and $u(t)$ is a r- dimensional, piecewise continuous (with a finite number of break points of the first kind) vector of control actions with values from the specified non-empty, bounded and open set$U \subset R^r$, i.e.,

$$u(t) \in U \subset R^r, t \in [t_0, t_1]. \tag{3}$$

Such control function will be called admissible controls.

It is assumed that there is a unique, continuous and piecewise smooth solution $x(t)$ of Cauchy problem (1), (2).

Consider the problem of the minimum of the terminal functional

$$S(u) = \varphi\big(x(T_1), x(T_2), \ldots, x(T_k)\big) + \int_{t_0}^{t_1}\int_{t_0}^{t_1} F(t,s,x(t),x(s))ds\,dt \tag{4}$$

with constraints (1)-(3).

Here, $T_i, i = \overline{1,k}$ $(t_0 < T_1 < T_2 < \cdots < T_k \le t_1)$ are specified points, $F(t,s,a,b)$ is a specified scalar function that is continuous in the set of variables together with partial derivatives in (a,b) up to and including the second order, $\varphi(c_1,\ldots,c_k)$ is a twice specified continuously differentiable scalar function.

The admissible control $u(t)$ that affords the minimum value of functional (4) under constraints (1)-(3) will be called an optimal control, and the corresponding process $\big(u(t),x(t)\big)$ an optimal process.

The aim of the article is to obtain first- and second-order necessary optimality conditions in the problem under investigation.

3. Calculating variations of the quality criterion

Suppose that $\big(u(t),x(t)\big)$ and $\big(\bar{u}(t) = u(t) + \Delta u(t), \bar{x}(t) = x(t) + \Delta x(t)\big)$ are two admissible processes.

The increment of the quality functional will be written as follows:

$$\Delta S(u) = S(\bar{u}) - S(u) = \varphi\big(\bar{x}(T_1), \bar{x}(T_2), \ldots, \bar{x}(T_k)\big) -$$

$$-\varphi\big(x(T_1), x(T_2), \ldots, x(T_k)\big) + \int_{t_0}^{t_1}\int_{t_0}^{t_1} \big(F(t,s,\bar{x}(t),\bar{x}(s)) - F(t,s,x(t),x(s))\big)ds\,dt. \tag{5}$$

It is clear that the increment $\Delta x(t)$ of the trajectory $x(t)$ is the solution to the problem

$$\Delta\dot{x}(t) = f\big(t,\bar{x}(t),\bar{u}(t)\big) - f(t,x(t),u(t)), \tag{6}$$

$$x(t_0) = 0. \tag{7}$$

Suppose that $\psi(t)$ is an as yet arbitrary n-dimensional vector function. We will introduce an analogue of the Hamilton-Pontryagin function in the following form:

$$H(t,x,u,\psi) = \psi' f(t,x,u).$$

Here and further in the text, the dash (') indicates a transpose operation.

From formula (6) we get that

$$\int_{t_0}^{t_1} \psi'(t)\Delta\dot{x}(t)dt = \int_{t_0}^{t_1} [H\big(t,\bar{x}(t),\bar{u}(t),\psi(t)\big) - H\big(t,x(t),u(t),\psi(t)\big)]dt. \tag{8}$$

Taking into account identity (8), from increment formula (5) of quality functional (4) we get that

$$\Delta S(u)=\varphi\big(\bar{x}(T_1),\bar{x}(T_2),\dots,\bar{x}(T_k)\big)-$$

$$-\varphi\big(x(T_1),x(T_2),\dots,x(T_k)\big)+\int_{t_0}^{t_1}\int_{t_0}^{t_1}\Big(F\big(t,s,\bar{x}(t),\bar{x}(s)\big)-F\big(t,s,x(t),x(s)\big)\Big)ds\,dt+$$

$$+\int_{t_0}^{t_1}\psi'(t)\Delta\dot{x}(t)dt-\int_{t_0}^{t_1}[H\big(t,\bar{x}(t),\bar{u}(t),\psi(t)\big)-H\big(t,x(t),u(t),\psi(t)\big)]dt. \qquad (9)$$

From formula (9) applying the Taylor formula to a separate summand, we get that

$$\Delta S(u)=\sum_{i=1}^{k}\frac{\partial\varphi'\big(x(T_1),x(T_2),\dots,x(T_k)\big)}{\partial c_i}\Delta x(T_i)+$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\Delta x'(T_i)\frac{\partial\varphi'\big(x(T_1),x(T_2),\dots,x(T_k)\big)}{\partial c_i}\Delta x\big(T_j\big)+o_1\left(\left[\sum_{i=1}^{k}\|\Delta x(T_i)\|\right]^2\right)+$$

$$+\int_{t_0}^{t_1}\psi'(t)\Delta x(t)dt-$$

$$-\int_{t_0}^{t_1}\left[\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial x}\Delta x(t)-\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial u}\Delta u(t)\right]dt-$$

$$-\frac{1}{2}\int_{t_0}^{t_1}\left[\Delta x'(t)\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial x^2}\Delta x(t)+\right.$$

$$\left.+2\,\Delta u'(t)\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial u\partial x}\Delta x(t)+\Delta u'(t)\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial u^2}\Delta u(t)\right]dt-$$

$$-\int_{t_0}^{t_1}o_2[(\|\Delta x(t)\|+\|\Delta u(t)\|)^2]dt+\int_{t_0}^{t_1}\int_{t_0}^{t_1}\frac{\partial F'\big(t,s,x(t),x(s)\big)}{\partial a}\Delta x(t)ds\,dt+$$

$$+\int_{t_0}^{t_1}\int_{t_0}^{t_1}\frac{\partial F'\big(t,s,x(t),x(s)\big)}{\partial b}\Delta x(s)ds\,dt+$$

$$+\frac{1}{2}\int_{t_0}^{t_1}\int_{t_0}^{t_1}\left[\Delta x'(t)\frac{\partial F\quad\big(t,s,x(t),x(s)\big)}{\partial a^2}\Delta x(t)+\right.$$

$$+\Delta x'(t)\frac{\partial F\big(t,s,x(t),x(s)\big)}{\partial a\partial b}\Delta x(s)+\Delta x'(s)\frac{\partial F\big(t,s,x(t),x(s)\big)}{\partial b\partial a}\Delta x(t)+$$

$$\left.+\Delta x'(s)\frac{\partial H\big(t,x(t),u(t),\psi(t)\big)}{\partial b^2}\Delta x(s)\right]dsdt+\int_{t_0}^{t_1}\int_{t_0}^{t_1}o_3([\|\Delta x(t)\|+\|\Delta x(s)\|]^2)ds\,dt\,. \qquad (10)$$

Clearly, by virtue of condition (7)

$$\Delta x(t) = \int_{t_0}^{t} \Delta\dot{x}(\tau)d\tau.$$

Therefore

$$\Delta x(T_i) = \int_{t_0}^{t_1} \alpha_i(t)\Delta\dot{x}(t)dt, \tag{11}$$

where $\alpha_i(t)$ is characteristic function of the segment$[t_0, T_i]$

Therefore, increment formula (10) can be written in the form

$$\Delta S(u) = \int_{t_0}^{t_1} \sum_{i=1}^{k} \alpha_i(t) \frac{\partial\varphi'\big(x(T_1), x(T_2), \dots, x(T_k)\big)}{\partial c_i} \Delta\dot{x}(t)dt +$$
$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta x'(T_i) \frac{\partial^2\varphi\big(x(T_1), x(T_2), \dots, x(T_k)\big)}{\partial c_i \partial c_j} \Delta x\big(T_j\big) + o_1\left(\left[\sum_{i=1}^{k} \|\Delta x(T_i)\|\right]^2\right) +$$
$$+\int_{t_0}^{t_1}\int_{t_0}^{t_1}\left[\int_{t}^{t_1} \frac{\partial F'\big(\tau, s, x(t), x(s)\big)}{\partial a} d\tau\right]\Delta\dot{x}(t)ds\,dt +$$
$$+\int_{t_0}^{t_1}\int_{t_0}^{t_1}\left[\int_{t}^{t_1} \frac{\partial F'\big(s, \tau, x(s), x(t)\big)}{\partial b} d\tau\right]\Delta\dot{x}(t)ds\,dt +$$
$$+\frac{1}{2}\int_{t_0}^{t_1}\int_{t_0}^{t_1}\Bigg[\Delta x'(t)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a^2}\Delta x(t) + \Delta x'(t)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a \partial b}\Delta x(s) +$$
$$+\Delta x'(s)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b \partial a}\Delta x(t) + \Delta x'(s)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b^2}\Delta x(s)\Bigg]dsdt +$$
$$+\int_{t_0}^{t_1}\int_{t_0}^{t_1} o_3([\|\Delta x(t)\| + \|\Delta x(s)\|]^2)ds\,dt +$$
$$+\int_{t_0}^{t_1} \psi'(t)\Delta\dot{x}(t)dt - \int_{t_0}^{t_1}\left[\int_{t}^{t_1} \frac{\partial H'\big(\tau, x(\tau), u(\tau), \psi(\tau)\big)}{\partial x} d\tau\right]\Delta\dot{x}(t)dt -$$
$$-\int_{t_0}^{t_1} \frac{\partial H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u}\Delta u(t)dt - \frac{1}{2}\int_{t_0}^{t_1}\Bigg[\Delta x'(t)\frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial x^2}\Delta x(t) +$$
$$+2\Delta u'(t)\frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u \partial x}\Delta x(t) + \Delta u'(t)\frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u^2}\Delta u(t) -$$
$$-\int_{t_0}^{t_1} o_2([\|\Delta x(t)\| + \|\Delta u(t)\|]^2)dt. \tag{12}$$

Suppose that $\psi(t)$ is the solution of the equation

$$\psi(t) = \int_t^{t_1} \frac{\partial H\big(\tau, x(\tau), u(\tau), \psi(\tau)\big)}{\partial x} d\tau - \sum_{i=1}^{k} \alpha_i(t) \frac{\partial \varphi'\big(x(T_1), x(T_2), \dots, x(T_k)\big)}{\partial c_i} -$$

$$- \int_{t_0}^{t_1} \int_t^{t_1} \frac{\partial F\big(\tau, s, x(\tau), x(s)\big)}{\partial a} d\tau\, ds - \int_{t_0}^{t_1} \int_t^{t_1} \frac{\partial F\big(s, \tau, x(s), x(\tau)\big)}{\partial b} d\tau\, ds. \quad (13)$$

Then increment formula (12) will take the form

$$\Delta S(u) = \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \Delta x'(T_i) \frac{\partial^2 \varphi\big(x(T_1), x(T_2), \dots, x(T_k)\big)}{\partial c_i \partial c_j} \Delta x(T_j) +$$

$$+ \frac{1}{2} \int_{t_0}^{t_1} \int_{t_0}^{t_1} \Bigg[\Delta x'(t) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a^2} \Delta x(t) + \Delta x'(t) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a \partial b} \Delta x(s) +$$

$$+ \Delta x'(s) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b \partial a} \Delta x(t) + \Delta x'(s) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b^2} \Delta x(s) \Bigg] ds dt -$$

$$- \int_{t_0}^{t_1} \frac{\partial H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u} \Delta u(t) dt - \frac{1}{2} \int_{t_0}^{t_1} \Bigg[\Delta x'(t) \frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial x^2} \Delta x(t) +$$

$$+ 2 \Delta u'(t) \frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u \partial x} \Delta x(t) + \Delta u'(t) \frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u^2} \Delta u(t) \Bigg] dt -$$

$$+ o_1\left(\left[\sum_{i=1}^{k} \|\Delta x(T_i)\| \right]^2 \right) + \int_{t_0}^{t_1} \int_{t_0}^{t_1} o_3([\|\Delta x(t)\| + \|\Delta x(s)\|]^2) ds\, dt -$$

$$- \int_{t_0}^{t_1} o_2([\|\Delta x(t)\| + \|\Delta u(t)\|]^2) dt. \quad (14)$$

Suppose that $\varepsilon > 0$ is an arbitrary sufficiently small in absolute value number, and$\delta u(t) \in U, t \in [t_0, t_1]$ is an arbitrary piecewise continuous and bounded *r*-dimensional vector function. Then, by virtue of the openness of the control domain U the special increment of the control $u(t)$ can be determined from the formula

$$\Delta u_\varepsilon(t) = \varepsilon\, \delta u(t), t \in [t_0, t_1]. \quad (15)$$

Suppose that $\Delta x_\varepsilon(t)$ U is the special increment of the trajectory $x(t)$ corresponding to the increment of the control $u(t)$.

From the estimates established, for instance, in [2, 4], it follows that

$$\|\Delta x(t)\| \le L \int_{t_0}^{t} \|\Delta u(\tau)\| d\tau, \qquad t \in [t_0, t_1],$$

where $L = const > 0$ is some constant.

From this we get that

$$\|\Delta x_\varepsilon(t)\| \le |\varepsilon| L \int_{t_0}^{t} \|\delta u(\tau)\| d\tau, \qquad t \in [t_0, t_1]. \quad (16)$$

Taking into account formulas (15) and (16) we prove the validity of the expansion

$$\Delta x_\varepsilon(t) = \varepsilon\,\delta x(t) + o(\varepsilon; t), \tag{17}$$

where $\delta x(t)$ (trajectory variation) is an *n*-dimensional vector function, which is the solution of the equation in variations [3]

$$\delta\dot{x}(t) = \frac{\partial f(t, x(t), u(t))}{\partial x}\delta x(t) + \frac{\partial f(t, x(t), u(t))}{\partial u}\delta u(t), \tag{18}$$

$$\delta x(t_0) = 0. \tag{19}$$

Taking into account formula (15) and decomposition (17), the special increment of the quality functional is as follows

$$S(u + \Delta u_\varepsilon) - S(u) = -\varepsilon \int_{t_0}^{t_1} \frac{\partial H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u}\delta u(t)dt +$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\delta x'(T_i)\frac{\partial^2\varphi\big(x(T_1), x(T_2), \dots, x(T_k)\big)}{\partial c_i \partial c_j}\delta x\big(T_j\big) +$$

$$+\frac{\varepsilon^2}{2}\int_{t_0}^{t_1}\int_{t_0}^{t_1}\left[\delta x'(t)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a^2}\delta x(t) + \delta x'(t)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a \partial b}\delta x(s) +\right.$$

$$\left.+\delta x'(s)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b \partial a}\delta x(t) + \delta x'(s)\frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b^2}\delta x(s)\right]dsdt -$$

$$-\frac{\varepsilon^2}{2}\int_{t_0}^{t_1}\left[\delta x'(t)\frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial x^2}\delta x(t) + 2\delta u'(t)\frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u \partial x}\delta x(t) +\right.$$

$$\left.+\,\delta u'(t)\frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u^2}\delta u(t)\right]dt + o(\varepsilon^2). \tag{20}$$

It follows from expansion (20) that the first and second variations (in the classical sense) of functional (4) respectively have the following form:

$$\delta^1 S(u\!: \delta u) = -\int_{t_0}^{t_1}\frac{\partial H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u}\delta u(t)dt, \tag{21}$$

$$\delta^2 S(u\!: \delta u) = \sum_{i=1}^{k}\sum_{j=1}^{k}\delta x'(T_i)\frac{\partial^2\varphi\big(x(T_1), x(T_2), \dots, x(T_k)\big)}{\partial c_i \partial c_j}\delta x\big(T_j\big) +$$

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1}\left[\delta x'(t)\frac{\partial^2 F'\big(t, s, x(t), x(s)\big)}{\partial a^2}\delta x(t) + \delta x'(t)\frac{\partial^2 F'\big(t, s, x(t), x(s)\big)}{\partial a \partial b}\delta x(s) +\right.$$

$$\left.+\delta x'(s)\frac{\partial^2 F'\big(t, s, x(t), x(s)\big)}{\partial b \partial a}\delta x(t) + \delta x'(s)\frac{\partial^2 F'\big(t, s, x(t), x(s)\big)}{\partial b^2}\delta x(s)\right]dsdt -$$

$$-\int_{t_0}^{t_1}\left[\delta x'(t)\frac{\partial^2 H'\big(t, x(t), u(t), \psi(t)\big)}{\partial x^2}\delta x(t) + 2\delta u'(t)\frac{\partial^2 H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u \partial x}\delta x(t) +\right.$$

$$\left.+\,\delta u'(t)\frac{\partial^2 H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u^2}\delta u(t)\right]dt + o(\varepsilon^2). \tag{22}$$

4. Necessary optimality conditions

Taking into account the main result of the calculus of variations (see, e.g., [3, 4]), we obtain from relations (21) and (22) that the optimality of the admissible control $u(t)$ requires that the relations

$$\int_{t_0}^{t_1} \frac{\partial H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u} \delta u(t) dt = 0, \tag{23}$$

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \delta x'(T_i) \frac{\partial^2 \varphi\big(x(T_1), x(T_2), \dots, x(T_k)\big)}{\partial c_i \partial c_j} \delta x\big(T_j\big) +$$

$$+ \int_{t_0}^{t_1}\int_{t_0}^{t_1} \Bigg[\delta x'(t) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a^2} \delta x(t) + \delta x'(t) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial a \partial b} \delta x(s) +$$

$$+ \delta x'(s) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b \partial a} \delta x(t) + \delta x'(s) \frac{\partial^2 F\big(t, s, x(t), x(s)\big)}{\partial b^2} \delta x(s)\Bigg] ds dt -$$

$$- \int_{t_0}^{t_1} \Bigg[\delta x'(t) \frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial x^2} \delta x(t) + 2\delta u'(t) \frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u \partial x} \delta x(t) +$$

$$+ \delta u'(t) \frac{\partial^2 H\big(t, x(t), u(t), \psi(t)\big)}{\partial u^2} \delta u(t)\Bigg] dt \geq 0 \tag{24}$$

hold for all $\delta u(t) \in R^r, t \in [t_0, t_1]$.

From identity (23) follows

Theorem 1. The optimality of the admissible control $u(t)$ requires that the relation

$$\frac{\partial H\big(\theta, x(\theta), u(\theta), \psi(\theta)\big)}{\partial u} = 0 \tag{25}$$

hold for all $\theta \in [t_0, t_1)$.

Here $\theta \in [t_0, t_1)$ is an arbitrary point of continuity of the control$u(t)$.

Necessary condition of optimality (25) being a first-order necessary optimality condition is an analogue of Euler's equation (see, e.g., [2, 6]) for the problem under investigation.

Each solution of the analogue of Euler's equation will be called a classical extremal. The number of classical extremals can be quite large (see, e.g., [5, 6]). Therefore, in order to "narrow down" the set of classical extremals, we must have second-order necessary optimality conditions.

Of course, inequality (24) is a second-order necessary optimality condition. But it is implicit.

We derive from it the second-order necessary optimality conditions of a constructive nature.

The solution of equation in variations (18)-(19) allows the following representation (see, e.g., [5]):

$$\delta x(t) = \int_{t_0}^{t} F(t, \tau) \frac{\partial f(\tau, x(\tau), u(\tau))}{\partial u} \delta u(\tau) d\tau, \tag{26}$$

where $F(t, \tau)$ is a $(n \times n)$ matrix function, which is the solution to the problem

$$F_\tau(t, \tau) = -F(t, \tau) \frac{\partial f\big(\tau, x(\tau), u(\tau)\big)}{\partial x},$$

$$F(t, t) = E,$$

where E is an identity matrix.

Introducing the notation

$$Q(t,\tau)=F(t,\tau)\frac{\partial f\big(\tau,x(\tau),u(\tau)\big)}{\partial u}$$

formula (26) is written in the form

$$\delta x(t)=\int_{t_0}^{t}Q(t,\tau)\delta u(\tau)d\tau. \qquad (27)$$

It follows from (27) that

$$\Delta x(T_i)=\int_{t_0}^{t_1}\alpha_i(t)Q(T_i,\tau)\delta u(\tau)d\tau. \qquad (28)$$

Using formula (28), we get that

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\delta x'(T_i)\frac{\partial^2\varphi\big(x(T_1),x(T_2),\dots,x(T_k)\big)}{\partial c_i\partial c_j}\delta x\big(T_j\big)=$$

$$=\int_{t_0}^{t_1}\int_{t_0}^{t_1}\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i(\tau)\alpha_i(s)\delta u'(\tau)Q'(T_i,\tau)\frac{\partial^2\varphi\big(x(T_1),x(T_2),\dots,x(T_k)\big)}{\partial c_i\partial c_j}Q\big(T_j,s\big)\,\delta u(s)dsd\tau. \qquad (29)$$

Further, using representation (27) according to the scheme similar to the scheme of [6, 7], we prove that

$$\int_{t_0}^{t_1}\Bigg[\delta x'(t)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial x^2}\delta x(t)+2\delta u'(t)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial u\partial x}\delta x(t)+$$

$$+\delta u'(t)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial u^2}\delta u(t)\Bigg]dt=$$

$$=\int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta u'(\alpha)\left(\int_{\max(\alpha,\beta)}^{t_1}Q'(t,\alpha)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial x^2}Q(t,\beta)dt\right)\delta u(\beta)d\alpha d\beta+$$

$$+2\int_{t_0}^{t_1}\left[\int_{t_0}^{t}\delta u'(t)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial u\partial x}Q(t,\alpha)\delta u(\alpha)d\alpha\right]dt+$$

$$+\int_{t_0}^{t_1}\delta u'(t)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial u^2}\delta u(t)dt, \qquad (30)$$

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1}\Bigg[\delta x'(t)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial a^2}\delta x(t)+\delta x'(t)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial a\partial b}\delta x(s)+$$

$$+\delta x'(s)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial b\partial a}\delta x(t)+\delta x'(s)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial b^2}\delta x(s)\Bigg]dsdt=$$

$$=\int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta u'(\alpha)\left[\int_{t_0}^{t_1}\left(\int_{\max(\alpha,\beta)}^{t_1}Q'(t,\alpha)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial a^2}Q(t,\beta)dt\right)ds+\right.$$

$$+\int_{\alpha}^{t_1}\left[\int_{\beta}^{t_1} Q'(t,\alpha)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial a\partial b}Q(s,\beta)ds\right]dt+$$

$$+\int_{\alpha}^{t_1}\left[\int_{\beta}^{t_1} Q'(s,\alpha)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial b\partial a}Q(s,\beta)dt\right]ds+$$

$$\left.+\int_{t_0}^{t_1}\left(\int_{\max(\alpha,\beta)}^{t_1} Q'(s,\alpha)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial b^2}Q(s,\beta)ds\right)dt\right]\delta u(\beta)d\alpha d\beta. \tag{31}$$

Introducing the notation

$$K(\alpha,\beta)=-\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i(\alpha)\alpha_i(\beta)Q'(T_i,\alpha)\frac{\partial^2\varphi\big(x(T_1),x(T_2),\dots,x(T_k)\big)}{\partial c_i\partial c_j}Q\big(T_j,\beta\big)+$$

$$+\int_{\max(\alpha,\beta)}^{t_1} Q'(t,\alpha)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial x^2}Q(t,\beta)dt-$$

$$-\int_{\max(\alpha,\beta)}^{t_1} Q'(t,\alpha)\frac{\partial^2 H\big(t,s,x(t),x(s)\big)}{\partial a^2}Q(t,\beta)dt-$$

$$-\int_{\alpha}^{t_1}\left[\int_{\beta}^{t_1} Q'(t,\alpha)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial a\partial b}Q(s,\beta)ds\right]dt-$$

$$-\int_{\alpha}^{t_1}\left[\int_{\beta}^{t_1} Q'(s,\alpha)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial b\partial a}Q(s,\beta)dt\right]ds-$$

$$-\int_{t_0}^{t_1}\left(\int_{\max(\alpha,\beta)}^{t_1} Q'(s,\alpha)\frac{\partial^2 F\big(t,s,x(t),x(s)\big)}{\partial b^2}Q(s,\beta)ds\right)dt$$

and taking into account identities (29)-(31), we get from inequality (24) that along the optimal control

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta u'(\alpha)\,K(\alpha,\beta)\delta u(\beta)d\alpha d\beta+$$

$$+2\int_{t_0}^{t_1}\left[\int_{t_0}^{t}\delta u'(t)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial u\partial x}Q(t,\alpha)\delta u(\alpha)d\alpha\right]dt+$$

$$+\int_{t_0}^{t_1}\delta u'(t)\frac{\partial^2 H\big(t,x(t),u(t),\psi(t)\big)}{\partial u^2}\delta u(t)dt\le 0. \tag{32}$$

Thus, we have proven

Theorem 2. The optimality of the classical extremal $u(t)$ requires that inequality (32) hold for all $\delta u(t)\in R^r,\alpha\in[t_0,t_1]$.

Inequality (32) is a rather general second-order necessary optimality condition.

From it, using the arbitrariness of $\delta u(t)$ we can obtain a number of relatively easily verifiable necessary optimality conditions.

For instance, an immediate corollary of Theorem 2 is

Theorem 3. The optimality of the classical extremal $u(t)$ requires that the inequality

$$v' \frac{\partial^2 H\big(\theta, x(\theta), u(\theta), \psi(\theta)\big)}{\partial u^2} v \le 0 \quad (33)$$

hold for all $v \in R^r$ and $\theta \in [t_0, t_1)$.

Inequality (33) is an analogue of the Legendre–Clebsch condition (see, e.g., [8]) for the problem under investigation.

We will study the case of its degeneration.

Definition. If for all $v \in R^r$ and $\theta \in [t_0, t_1]$

$$v' \frac{\partial^2 H\big(\theta, x(\theta), u(\theta), \psi(\theta)\big)}{\partial u^2} v = 0 \quad (34)$$

then the classical extremal $u(t)$ will be called a classically singular control in the problem under investigation.

From inequality (32), determining the admissible variation $\delta u(t)$ of the control $u(t)$ in a special manner, we get that the following inequality holds along the classically singular optimal control $u(t)$:

$$v'(K(\theta,\theta)v + H_{ux}\big(\theta, x(\theta), u(\theta), \psi(\theta)\big))v \le 0. \quad (35)$$

Theorem 4. It is necessary for the classical optimality of the control $u(t)$ that inequality (35) hold for all $v \in R^r$ and $\theta \in [t_0, t_1)$.

Note that inequality (35) is an analogue of the Gabasov-Kirillova optimality condition obtained by them in [8] for a simpler problem in a completely different way.

5. Conclusion

A non-type optimal control problem with a non-type multipoint quality functional is considered. Taking into account the specifics of the problem, an adjoint system is introduced in the form of an integral equation.

The first and second variations of the functional are calculated, which allow us to formulate the first- and second-order necessary optimality conditions, explicitly expressed through the parameters of the problem under investigation.

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