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Sweep method for solution of linear quadratic optimization problem with constraint in the form of equalities on control

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A B S T R A C T

In the article the linear quadratic optimization problem with boundary condition is investigated. Here in a certain part of time interval some coordinates of control action are known constants. An extended functional corresponding to the given quadratic functional is constructed and the Euler-Lagrange equation with boundary conditions is obtained. Using sweep method the given linear quadratic optimization problem has been solved and the control actions and the corresponding program trajectory on each part of the time interval have been constructed. The results are illustrated using the example of the vertical motion of a flying vehicle.

1. Introduction

Recently, an important role has been played by the construction of program trajectories and controls during the movement of a flying vehicle [1-5]. However, it is of great interest when a flying vehicle, during vertical movement, at what point in altitude should stop and stand for a certain time. Therefore, for an ordinary linear quadratic problem (LQP) [6-8], control must have restrictions in the form of equalities on parts of the control actions. Therefore, there is a linear quadratic control problem with equality constraints. This problem is an addition to the similar linear quadratic control problem [9-12]. Note that this is a problem with an increased dimension of the given problem. However, to use solutions of this problem on a specific example, it makes sense to consider using the sweep method, which has significantly less dimension.

In the current article, the sweep method is developed for constructing program trajectories [13, 14]. Note that such an algorithm helps to find the missing initial conditions and reduce it to a

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solution of a system of linear algebraic equations (SLAE), where the main matrix is symmetric, which provides good conditionality. Here this SLAE requires the solution of the corresponding matrix differential equations of Riccati, Lyapunov, where it is well developed in MATLAB [15]. The results are illustrated on the example of the vertical movement of a quadcopter and coincide with known results at a certain accuracy.

2. Problem statement

Let the movement of the object in the time intervals $[0,T),\ (T,T+\Delta]$ is described by the following linear differential equation

$$\dot{x}(t) = Fx(t) + Gu(t) + V, \tag{1}$$

with boundary conditions for the phase coordinates

$$Hx(0) = q. \tag{2}$$

Assume that there is a condition at the time instant T

$$x(T+0) = F_\delta x(T-0) + G_1 u_1(T-0), \tag{3}$$

where $x(t)-$ phase coordinates of n-dimensional object, F, G and H are constant matrices with dimensions $n\times n,\ n\times m,\ m\times n,$ respectively, q is m dimensional known vector. The sign $'$ means the transpose operation. V is an external disturbance with dimension $n\times 1,\ u'(t) = [u_1(t) \quad u_2(t)]-$ m-dimensional piecewise continuous vector of control actions from R^m Euclidean space, $u_1(t)$, $u_2(t)$ are $(m\text{-}p)$ and p – dimensional vectors, correspondingly, Δ- is the time when the control is stable for the object, F_δ, G_1 are known matrices with corresponding dimensions.

It is required to find control action $u(t)$ and the corresponding program trajectory $x(t)$ with boundary conditions (2), (3) such that the equation (1) is satisfied and the following quadratic functional received the minimum value

$$\begin{aligned} J &= \frac{1}{2}(x(T-0)-x_d)'\overline{N}(x(T-0)-x_d) + \frac{1}{2}(u(T-0)-C)'\delta(u(T-0)-C) + \\ &+ \frac{1}{2}\int_0^T (x'(t)Qx(t) + 2x'(t)Ku(t) + u'(t)Ru(t))dt + \frac{1}{2}\int_T^{T+\Delta}(x'(t)\overline{Q}x(t) + (u(t)-C)'\gamma(u(t)-C)dt, \end{aligned} \tag{4}$$

where $\overline{N} \ge 0,\ \delta = \delta' > 0,\ R = R' > 0,\ Q = Q' \ge 0,\ \overline{Q} = \overline{Q}' \ge 0,\ \gamma = \gamma' > 0,\ K,\ \ x_d, C$ are known matrices and vectors with corresponding dimensions. Note that $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$, C_1 is a known constant parameter.

3. Sweep method for the problem (1)-(4)

Constructing the extended functional, the problem (1)-(4) is reduced the following boundary value problem

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} F - GR^{-1}K' & -GR^{-1}G' \\ -Q' + KR^{-1}K' & -F' + KR^{-1}G' \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} V \\ 0 \end{bmatrix},\ 0 \le t < T, \tag{5}$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} F & -G\gamma^{-1}G' \\ 0 & -F' \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} V+GC \\ 0 \end{bmatrix}, \; T < t \le T+\Delta, \tag{6}$$

with boundary conditions

$$\overline{N}'x(T-0) + \overline{N}'x_d - \lambda(T-0) + F_\delta'\lambda(T+0) = 0, \tag{7}$$

$$\delta u(T-0) - \delta C + G_1'\lambda(T+0) = 0, \tag{8}$$

$$\lambda(T+\Delta) = 0, \tag{9}$$

$$H'\nu + \lambda(0) = 0, \tag{10}$$

where $\lambda(t)$, ν are Lagrange multipliers with corresponding dimensions.

The control $u(t)$ on the intervals $[0,T)$ and $(T, T+\Delta]$ are defined as follows, respectively

$$u(t) = -R^{-1}K'x(t) - R^{-1}G'\lambda(t), \tag{11}$$

$$u(t) = C - \gamma^{-1}G'\lambda(t). \tag{12}$$

Using the linearity of the system of equations (5) and (6), we assume that $\lambda(t)$ is a linear function of $x(t)$ [13, 14], i.e.

$$\lambda(t) = S(t)x(t) + N(t)\nu + \omega(t), \tag{13}$$

where the unknown matrices $S(t)$, $N(t)$ and vector $\omega(t)$ are the corresponding dimensions. Substituting (13) into the first equations of (5) and (6) (taking into account the constancy of the vector ν), after some transformations we get that $S(t)$, $N(t)$, $\omega(t)$ satisfy the following Riccati, matrix and vector differential equations [15], respectively

$$\begin{cases} \dot{S}(t) = -\overline{F}'S(t) - S(t)\overline{F} + S(t)\overline{M}S(t) - \overline{R}, \\ \dot{N}(t) = (S(t)\overline{M} - \overline{F}')N(t), & 0 \le t < T \\ \dot{\omega}(t) = (S(t)\overline{M} - \overline{F}')\omega(t) - S(t)V, \end{cases} \tag{14}$$

$$\begin{cases} \dot{S}(t) = -F'S(t) - S(t)F + S(t)MS(t) - \overline{Q}', \\ \dot{N}(t) = (S(t)M - F')N(t), & T < t \le T+\Delta \\ \dot{\omega}(t) = (S(t)M - F')\omega(t) - S(t)\overline{V}, \end{cases} \tag{15}$$

where

$$\overline{F} = GR^{-1}K', \; \overline{M} = GR^{-1}G', \; \overline{R} = Q' - KR^{-1}K', M = G\gamma^{-1}G', \; \overline{V} = V + GC.$$

From (9) we get that

$$S(T+\Delta) = 0, \; N(T+\Delta) = 0, \; \omega(T+\Delta) = 0. \tag{16}$$

Integrating equations (15) with initial conditions (16) in reverse time from moment $T+\Delta$ to T and taking into account the conditions at the time instant T using (3), (7), (8), i.e.

$$\begin{aligned} S(T-0) &= \left((F_\delta')^{-1} + S(T+0)G_1\delta^{-1}G_1'(F_\delta')^{-1}\right)^{-1} \times \\ &\times \left(S(T+0)F_\delta + (F_\delta')^{-1}\overline{N}' + S(T+0)G_1\delta^{-1}G_1'(F_\delta')^{-1}\overline{N}'\right), \\ N(T-0) &= \left((F_\delta')^{-1} + S(T+0)G_1\delta^{-1}G_1'(F_\delta')^{-1}\right)^{-1} N(T+0), \end{aligned} \tag{17}$$

$$\omega(T-0)=\left((F_\delta')^{-1}+S(T+0)G_1\delta^{-1}G_1'(F_\delta')^{-1}\right)^{-1}\times$$
$$\times\left(S(T+0)G_1C-S(T+0)G_1\delta^{-1}G_1'(F_\delta')^{-1}\bar{N}'x_d+\omega(T+0)-(F_\delta')^{-1}\bar{N}x_d\right),$$

then integrating equations (14) with initial conditions (17) in reverse time from moment T to zero, from (2) and (13) we obtain

$$S(0)x(0)+(N(0)+H')\nu+\omega(0)=0. \tag{18}$$

On the other hand, using the results [14], it can be shown that

$$(N'(0)+H)x(0)+n(0)\nu=q-W(0), \tag{19}$$

where the matrix $n(t)$ and vector $W(t)$ are defined by the following differential equations, respectively:

$$\begin{cases}\dot{n}(t)=N'(t)\bar{M}N(t), \quad n(T-0)=0,\\ \dot{W}(t)=N'(t)(\bar{M}\omega(t)-V), \ W(T-0).\end{cases}$$

From (18) and (19) we get the system of linear algebraic differential equations:

$$\begin{bmatrix} S(0) & N(0)+H' \\ N'(0)+H & n(0)\end{bmatrix}\begin{bmatrix} x(0) \\ \nu \end{bmatrix}=\begin{bmatrix} -\omega(0) \\ q-W(0)\end{bmatrix}. \tag{20}$$

Further, we can find the optimal program trajectory $x(t)$ solving the following differential equation on the interval [0,*T*) by the initial condition $x(0)$

$$\dot{x}(t)=(F-GR^{-1}K'-GR^{-1}G'S(t))x(t)-GR^{-1}G'(N(t)\nu+\omega(t))+V. \tag{21}$$

Then we find the optimal program trajectory $x(t)$ on the interval (*T*, *T*+Δ] by the initial condition $x(T+0)$ obtaining (3)

$$\dot{x}(t)=(F-G\gamma^{-1}G'S(t))x(t)-G\gamma^{-1}G'(N(t)\nu+\omega(t))+V+GC.$$

And the control *u*(*t*) on the intervals [0,*T*) and [*T*, *T*+Δ] are defined as follows, respectively

$$u(t)=-R^{-1}[(K'+G'S(t))x(t)+G'N(t)\nu+G'\omega(t)], \tag{22}$$

$$u(t)=C-\gamma^{-1}G'(S(t)x(t)+N(t)\nu+\omega(t)). \tag{23}$$

Note that *u*(*T-0*) is defined from (8).

4. Example

Let us consider an example that describes the vertical movements of a flying object on the interval [*0*, *T*+Δ]. For simplicity, we will assume that the object moves only vertically and does not perform any rotation relative to the vertical axis. In this case, from (1) and (2)

$x(t)=\begin{bmatrix} z_1(t) \\ z_2(t)\end{bmatrix}$, $F=\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$, $G=\begin{bmatrix} 0 \\ 1\end{bmatrix}$, $V=\begin{bmatrix} 0 \\ -g\end{bmatrix}$, g is a gravitational acceleration, $H=\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$, $q=z_{10}=0$.

Now introduce the following functional which is to be minimized

$$J = \frac{1}{2}(x(T-0) - x_d)'\overline{N}(x(T-0) - x_d) + \frac{1}{2}(u(T-0) - c)^2 \delta +$$
$$+\frac{1}{2}\int_0^T (x'(t)Qx(t) + 2x'(t)Ku(t) + Ru^2(t))dt + \frac{1}{2}\int_T^{T+\Delta} (x'(t)\overline{Q}x(t) + \gamma(u(t) - c)^2 \gamma)dt,$$

where

$$N = \begin{bmatrix} \chi & 0 \\ 0 & \alpha \end{bmatrix}, \; x_d = \begin{bmatrix} z_d \\ 0 \end{bmatrix}, \; Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \; K = \begin{bmatrix} k_1 \\ 0 \end{bmatrix}, \; \overline{Q} = \begin{bmatrix} \overline{q}_1 & 0 \\ 0 & \overline{q}_2 \end{bmatrix},$$

$\chi = 10^{14}$, $\alpha = 10^6, z_d = 2, \delta = 10^9, c = 9.8$, $R = 10^{10}$, $q_1 = 10, q_2 = 190, k_1 = -10^{11}$,

$\overline{q}_1 = 10, \overline{q}_2 = 10^{10}$, $\gamma = 10^{15}$.

Using the procedure outlined in sections 2-3, the required problem has been solved.

Let us introduce the graphs of optimal program trajectory $z_1(t)$ and the control action $u(t)$ on the interval [0, T+Δ].

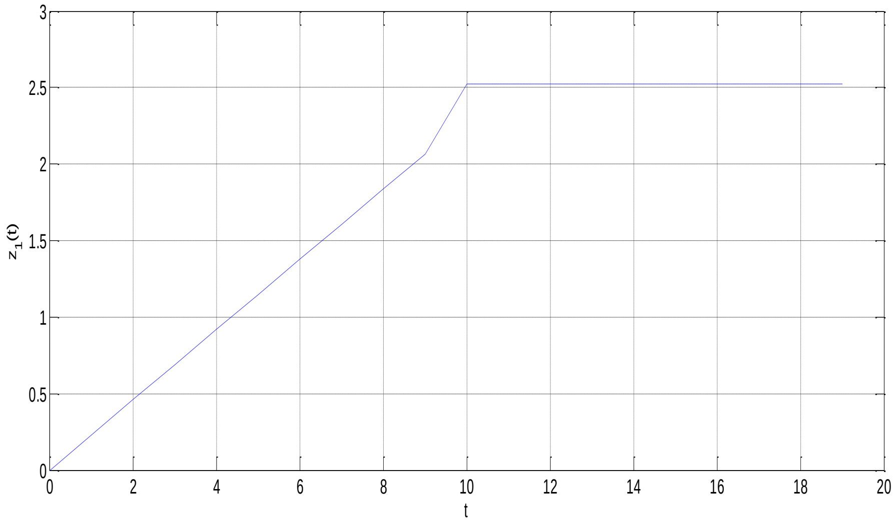

Fig.1. Changing $z_1(t)$ on the interval [0, T+Δ]

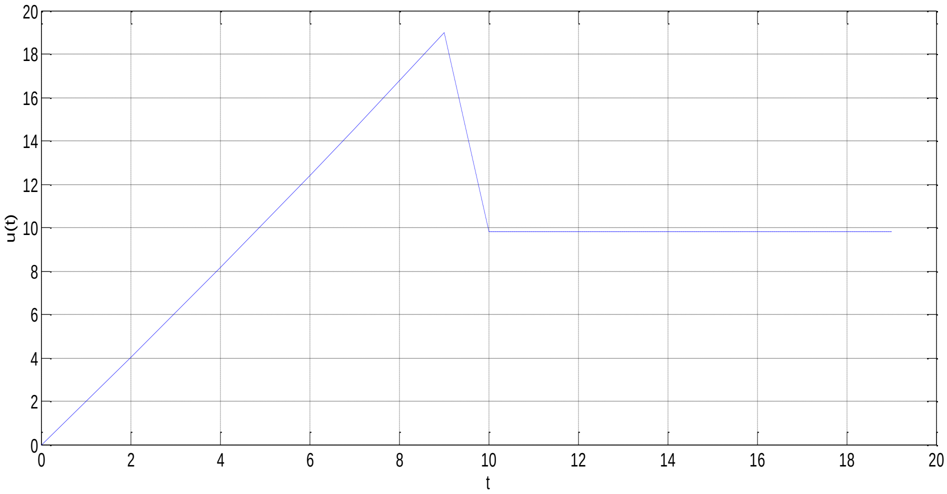

Fig.2. Changing $u(t)$ on the interval [0, T+Δ]

5. Conclusion

In this work, we consider a linear quadratic optimization problem, where in a known part of the time interval some coordinates of control action are known. Using sweep method the problem is reduced linear matrix and nonlinear Riccati equations. Solving these equations we introduce the graphs of control actions and the program trajectory for each part of the time interval. The discrete formulation of this problem is also important and it can be used to find the optimal program trajectory and control of aircraft.

References

[1] V.B. Larin, A.A. Tunik, S.I. Ilnytska, Some algorithms for unmanned aerial vehicles navigation, Outskirts, (2020) 204 p.

[2] M.M. Mutallimov, N.S. Hajiyeva, Construction of optimal program trajectories and controls for vertical movement of a quadcopter, Proceedings of IAM, 12 No.2 (2023) pp.166-180. [In Russian].

[3] F.A. Aliev, M.M. Mutallimov, N.I. Velieva, N.A. Safarova, A.G. Jafarov, Algorithms for solving the stabilization problems of the quadcopter motion by GPS data, Applied and Computational Mathematics. 22 No.4 (2023) pp.520-527.

[4] F.A. Aliev, G.N. Javadov, M.M. Mutallimov, Problems of stabilizing the movement of a quadcopter according to GPS data, ANAS, Reports. 79 No.1-2 pp.10-16. [In Russian].

[5] F.A. Aliev, O.A. Sushchenko, M.M. Mutallimov, N.G. Javadov, F.F. Mammadov, R.R. Maharramov, Algorithm for quadcopter motion stabilization taking into account data of inertial navigation system, TWMS J. Pure Appl. Math. 14 No.2 (2023) pp.278-289.

[6] F.A. Aliev, A.P. Guliev, A.N. Abbasov, Calculation algorithm of optimization problem two-point boundary conditions, Proc.of 26th Annual Iranian Math. Conference, Kerman, Iran, (1995).

[7] F.A. Aliev, V.B. Larin, Optimization of Linear Control System, London, Gordon and Breach, (1998) 270 p.

[8] F.A. Aliev, Methods for solving applied problems of optimization of dynamic systems, Baku, Elm, (1989) 320 p. [In Russian].

[9] F.A. Aliev, Optimal control problem for a linear system with nonseparated two-point boundary conditions, Differential equations. No.2 (1986) pp. 345-347. [In Russian].

[10] F.A. Aliev, Optimization problem with two-point boundary conditions, News of the USSR Academy of Sciences, ser.techn. Cybernetics. No.6 (1985) pp.138-146. [In Russian].

[11] H. Kvakernaak, R. Sivan, Linear optimal control systems, Moscow, Mir, (1977) 656 p. [In Russian].

[12] F.A. Aliev, V.B. Larin, K.I. Naumenko, V.N. Suntsev, Optimization of linear time-invariant control systems, Kyiv, Naukova Dumka, (1978) 327 p. [In Russian].

[13] A. Bryson, Ho Yu Shi, Applied theory of optimal control, Moscow, Mir, (1972). [In Russian].

[14] E. Polak, Numerical optimization methods, Unified approach, Moscow, Mir, (1971) 424 p. [In Russian].

[15] F.A. Aliev, B.A. Bordyug, V.B. Larin, Spectral method for solving matrix algebraic Riccati equations, Dokl. AN USSR. 292 No.4 (1987) pp.783-788. [In Russian].