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An analogue of the Pontryagin maximum principle in a discrete-continuous stepwise control problem

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ABSTRACT

The article considers one stepwise discrete-continuous optimal control problem described by a set of Volterra difference equations and Volterra-type integro-differential equations. An analogue of L.S. Pontryagin's maximum principle (discrete-continuous maximum principle) is proved.

1. Introduction

Various stepwise optimal control problems described by systems of differential equations have been studied in [1-4] and others.

In [5], one step-wise optimal control problem described by systems of Volterra difference equations and Volterra integro-differential equations is considered.

In this article, we consider a more general problem than in [5] and prove an analogue of L.S. Pontryagin's maximum principle (see, e.g., [6-8]) (discrete-continuous maximum principle).

2. Problem statement

Suppose that a controlled process is described by a set of Volterra difference equations and Volterra integro-differential equations

$$x_1(t+1) = f_1\big(t, x_1(t), u_1(t)\big) + \\ + \sum_{\tau=t_0}^{t} K_1\big(t, \tau, x_1(\tau), u_1(\tau)\big), t \in T_1 = \{t_0, t_0+1, \dots, t_1-1\}, \tag{1}$$

$$x_1(t_0) = x_{10}, \tag{2}$$

$$\dot{x}_2(t) = f_2\big(t, x_2(t), u_2(t)\big) + \\ + \int_{t_1}^{t} K_2\big(t, \tau, x_2(\tau), u_2(\tau)\big) d\tau, t \in T_2 = [t_1, t_2], \tag{3}$$

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$$x_2(t_1) = G\big(x_1(t_1)\big). \tag{4}$$

Here $f_1(t, x_1, u_1)$ $\big(K_1(t, \tau, x_1, u_1)\big)$ is a prescribed n-dimensional vector function discrete in t (t, τ) and continuous in (x_1, u_1) together with partial derivatives of x_1, $f_2(t, x_2, u_2), K_2(t, \tau, x_2, u_1)$ are prescribed n-dimensional vector functions continuous in the set of variables together with partial derivatives of x_2, x_{10} is a prescribed n-dimensional continuous vector, $G(x_1)$ is a prescribed continuous-differential $n$dimensional vector-function, $u_1(t)$ $\big(u_2(t)\big)$ is a $r(q)$-dimensional discrete (piecewise continuous, with a finite number of points of discontinuity of the first kind) vector function of control actions, with values from a non-empty and bounded set U_1 (U_2) , i.e.

$$u_1(t) \in U_1 \subset R^r, t \in T_1,$$
$$u_2(t) \in U_2 \subset R^q, t \in T_2. \tag{5}$$

The pair $(u_1(t), u_2(t))$ with the above properties will be called an admissible control.

We will assume that the given admissible control $\big(u_1(t), u_2(t)\big)$ has the unique discrete solution $x_1(t)$ of problem (1)-(2) and the unique piecewise smooth solution $x_2(t)$ of problem (3)-(4).

On the solutions of Cauchy problems (1)-(4) generated by all possible admissible controls we define the functional

$$J(u_1, u_2) = \varphi_1\big(x_1(t_1)\big) + \sum_{t=t_0}^{t_1-1} \sum_{\tau=t_0}^{t} F_1\big(t, \tau, x_1(\tau), u_1(\tau)\big) + \varphi_2\big(x_2(t_2)\big) +$$
$$+ \int_{t_1}^{t_2} \left[\int_{t_1}^{t} F_2\big(t, \tau, x_2(\tau), u_2(\tau)\big) d\tau \right] dt. \tag{6}$$

Here $\varphi_1(x_1), \varphi_2(x_2)$ are prescribed continuously differentiable scalar functions, $F_1(t, \tau, x_1, u_1)$ is a prescribed scalar function discrete in (t, τ) and continuously differentiable in x_1, and $F_2(t, \tau, x_2, u_2)$ is a prescribed scalar function continuous over the set of variables together with partial derivatives of x_2.

Consider the problem of finding the minimum value of functional (6) under constraints (1)-(5).

The admissible control $\big(u_1(t), u_2(t)\big)$ that affords the minimum value to functional (6) under constraints (1)-(5) will be called optimal control, and the corresponding process $\big(u_1(t), u_2(t), x_1(t), x_2(t)\big)$ an optimal process.

As can be seen, the optimal control problem under consideration is a Boltza problem.

The aim of the article is to derive a necessary optimality condition in the problem under consideration.

3. Formula of the increment of the quality functional

Suppose that $\big(u_1(t), u_2(t), x_1(t), x_2(t)\big)$ and $\big(\bar{u}_1(t) = u_1(t) + \Delta u_1(t),\ \bar{u}_2(t) = u_2(t) + \Delta u_2(t), \bar{x}_1(t) = x_1(t) + \Delta x_1(t), \bar{x}_2(t) = x_2(t) + \Delta x_2(t)\big)$ are some admissible processes.

We write down the increment of the quality functional:

$$\Delta J(u_1, u_2) = J(\bar{u}_1, \bar{u}_2) - J(u_1, u_2) = \varphi_1\big(\bar{x}_1(t_1)\big) - \varphi_1\big(x_1(t_1)\big) + \varphi_2\big(\bar{x}_2(t_2)\big) - \varphi_2\big(x_2(t_2)\big) +$$

$$+\sum_{t=t_0}^{t_1-1}\left[\sum_{\tau=t_0}^{t}\left[F_1\big(t,\tau,\bar{x}_1(\tau),\bar{u}_1(\tau)\big)-F_1\big(t,\tau,x_1(\tau),u_1(\tau)\big)\right]\right]+$$
$$+\int_{t_1}^{t_2}\left[\int_{t_1}^{t}\left[F_2\big(t,\tau,\bar{x}_2(\tau),\bar{u}_2(\tau)\big)-F_2\big(t,\tau,x_2(\tau),u_2(\tau)\big)\right]d\tau\right]dt. \tag{7}$$

It is clear that $\big(\Delta x_1(t),\Delta x_2(t)\big)$ is the solution to the problem

$$\Delta x_1(t+1)=f_1\big(t,\bar{x}_1(t),\bar{u}_1(t)\big)-f_1\big(t,x_1(t),u_1(t)\big)+$$
$$+\sum_{\tau=t_0}^{t}\left[K_1\big(t,\tau,\bar{x}_1(\tau),\bar{u}_1(\tau)\big)-K_1\big(t,\tau,x_1(\tau),u_1(\tau)\big)\right], \tag{8}$$
$$\Delta x_1(t_0)=0, \tag{9}$$
$$\Delta\dot{x}_2(t)=f_2\big(t,\bar{x}_2(t),\bar{u}_2(t)\big)-f_2\big(t,x_2(t),u_2(t)\big)+$$
$$+\int_{t_1}^{t}\left[K_2\big(t,\tau,\bar{x}_2(\tau),\bar{u}_2(\tau)\big)-K_2\big(t,\tau,x_2(\tau),u_2(\tau)\big)\right]d\tau, \tag{10}$$
$$\Delta x_2(t_1)=G\big(\bar{x}_1(t)\big)-G\big(x_1(t_1)\big). \tag{11}$$

Let $\psi_i=\psi_i(t), i=1{,}2$ be as yet arbitrary -dimensional vector functions.

From identities (8) and (10), using a discrete analogue of the Fubini theorem (see, e.g., [9]) and the Fubini formula (see, e.g., [7]) after some transformations we obtain the validity of the relations

$$\psi_1'(t_1-1)\Delta x_1(t_1)+\sum_{t=t_0}^{t_1-1}\psi_1{}'(t-1)\Delta x_1(t)=$$
$$=\sum_{t=t_0}^{t_1-1}\psi_1{}'(t)\left(f_1\big(t,\bar{x}_1(t),\bar{u}_1(t)\big)-f_1\big(t,x_1(t),u_1(t)\big)\right)+$$
$$+\sum_{t=t_0}^{t_1-1}\left[\sum_{\tau=t}^{t_1-1}\psi_1'(\tau)\left(K_1\big(\tau,t,\bar{x}_1(t),\bar{u}_1(t)\big)-K_1\big(\tau,t,x_1(t),u_1(t)\big)\right)\right]. \tag{12}$$

$$\psi_2{}'(t_2)\Delta x_2(t_2)-\psi_2{}'(t_1)\left(G\big(\bar{x}_1(t_1)\big)-G\big(x_1(t_1)\big)\right)-\int_{t_1}^{t_2}\dot{\psi}_2{}'(t)\Delta x_2(t)dt=$$
$$=\int_{t_1}^{t_2}\psi_2{}'(t)\left(f_2\big(t,\bar{x}_2(t),\bar{u}_2(t)\big)-f_2\big(t,x_2(t),u_2(t)\big)\right)dt+$$
$$+\int_{t_1}^{t_2}\left[\int_{t}^{t_2}\psi_2{}'(\tau)\left[K_2\big(\tau,t,\bar{x}_2(t),\bar{u}_2(t)\big)-K_2\big(\tau,t,x_2(t),u_2(t)\big)\right]d\tau\right]dt. \tag{13}$$

Let us introduce analogues of the Hamilton-Pontryagin function in the form

$$H_1\big(t, x_1(t), u_1(t), \psi_1(t)\big) = \psi_1'(t) f_1\big(t, x_1(t), u_1(t)\big) +$$
$$+\sum_{\tau=t}^{t_1-1}\left[\psi_1'(\tau) K_1\big(\tau, t, x_1(t), u_1(t)\big) - F_1\big(\tau, t, x_1(t), u_1(t)\big)\right], \tag{14}$$
$$H_2\big(t, x_2(t), u_2(t), \psi_2(t)\big) = \psi_2'(t) f_2\big(t, x_2(t), u_2(t)\big) +$$
$$+\int_t^{t_2}\left[\psi_2{}'(\tau) K_2\big(\tau, t, x_2(t), u_2(t)\big) - F_2\big(\tau, t, x_2(t), u_2(t)\big)\right] d\tau, \tag{15}$$
$$M(\psi_2(t_1), x_1(t_1)) = \psi_2{}'(t_1) G\big(x_1(t_1)\big). \tag{16}$$

Taking into account identities (12), (13) and notations (14)-(16), increment (7) of the quality functional is represented as

$$\Delta J(u_1, u_2) = \varphi_1\big(\bar{x}_1(t_1)\big) - \varphi_1\big(x_1(t_1)\big) + \varphi_2\big(\bar{x}_2(t_2)\big) - \varphi_2\big(x_2(t_2)\big) -$$
$$-\sum_{t=t_0}^{t_1-1}\left[H_1\big(t, \bar{x}_1(t), \bar{u}_1(t), \psi_1(t)\big) - H_1\big(t, x_1(t), u_1(t), \psi_1(t)\big)\right] + \psi_1'(t_1-1)\Delta x_1(t_1) +$$
$$+\sum_{t=t_0}^{t_1-1}\psi_1{}'(t-1)\Delta x_1(t) - \Big(M\big(\psi_2(t_1), \bar{x}_1(t_1)\big) - M\big(\psi_2(t_1), x_1(t_1)\big)\Big) +$$
$$+\psi_2{}'(t_2)\Delta x_2(t_2) - \int_{t_1}^{t_2}\dot{\psi}_2{}'(t)\Delta x_2(t) dt -$$
$$-\int_{t_1}^{t_2}\left[H_2\big(t, \bar{x}_2(t), \bar{u}_2(t), \psi_2(t)\big) - H_2\big(t, x_2(t), u_2(t), \psi_2(t)\big)\right] dt. \tag{17}$$

Let us introduce the notations

$$\Delta_{\bar{u}_i} H_i[t, x_i, \psi_i] \equiv H_i\big(t, x_i(t), \bar{u}_i(t), \psi_i(t)\big) - H_i\big(t, x_i(t), u_i(t), \psi_i(t)\big), i = 1,2,$$

$$\Delta_{\bar{x}_i} H_i[t, x_i, \psi_i] \equiv H_i\big(t, \bar{x}_i(t), u_i(t), \psi_i(t)\big) - H_i\big(t, x_i(t), u_i(t), \psi_i(t)\big), i = 1,2,$$
$$\frac{\partial H_i[t, x_i \psi_i]}{\partial x_i} \equiv \frac{\partial H_i\big(t, x_i(t), u_i(t), \psi_i(t)\big)}{\partial x_i}, i = 1,2,$$
$$\Delta_{\bar{u}_i} f_i\,[t] \equiv f_i\,\big(t, x_i(t), \bar{u}_i(t)\big) - f_i\,\big(t, x_i(t), u_i(t)\big), i = 1,2,$$
$$\frac{\partial f_i\,[t]}{\partial x_i} \equiv \frac{\partial f_i\,\big(t, x_i(t), u_i(t)\big)}{\partial x_i}, i = 1,2.$$

Taking into account the introduced notations and the Taylor formula, increment (17) of functional (6) is represented in the form

$$\Delta J(u_1,u_2)=\frac{\partial\varphi_1'\big(x_1(t_1)\big)}{\partial x_1}\Delta x_1(t_1)+o_1(\|\Delta x_1(t_1)\|)+\frac{\partial\varphi_2'\big(x_2(t_2)\big)}{\partial x_2}\Delta x_2(t_2)+o_2(\|\Delta x_2(t_2)\|)+$$
$$+\psi_1'(t_1-1)\Delta x_1(t_1)+\sum_{t=t_0}^{t_1-1}\psi_1'(t-1)\Delta x_1(t)-\sum_{t=t_0}^{t_1-1}\Delta_{\bar u_1}H_1[t,x_1,\psi_1]-\sum_{t=t_0}^{t_1-1}\frac{\partial H_1{}'[t,x_1,\psi_1]}{\partial x_1}\Delta x_1(t)-$$
$$-\sum_{t=t_0}^{t_1-1}\frac{\partial\Delta_{\bar u_1}H_1{}'[t,x_1,\psi_1]}{\partial x_1}\Delta x_1(t)-\sum_{t=t_0}^{t_1-1}o_3(\|\Delta x_1(t)\|)-$$
$$-\frac{\partial M'\big(\psi_2(t_1),x_1(t_1)\big)}{\partial x_1}\Delta x_1(t_1)-o_4(\|\Delta x_1(t_1)\|)+$$
$$+\psi_2'(t_2)\Delta x_2(t_2)-\int_{t_1}^{t_2}\dot\psi_2{}'(t)\Delta x_2(t)\,dt-\int_{t_1}^{t_2}\Delta_{\bar u_2}H_2[t,x_2,\psi_2]dt-$$
$$-\int_{t_1}^{t_2}\frac{\partial H_2{}'[t,x_2,\psi_2]}{\partial x_2}\Delta x_2(t)dt-\int_{t_1}^{t_2}\frac{\partial\Delta_{\bar u_2}H_2{}'[t,x_2,\psi_2]}{\partial x_2}\Delta x_2(t)dt-\int_{t_1}^{t_2}o_5(\|\Delta x_2(t)\|)dt. \quad (18)$$

Here, and in the following, $\|\alpha\|$ is the norm of the vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)'$, in linear - dimensional space R^n defined by the formula $\|\alpha\|=\sum_{i=1}^{n}|\alpha_i|$, and $o(\alpha)$ is a quantity having a higher order of smallness than α, i.e. $\frac{o(\alpha)}{\alpha}\to 0$ at $\alpha\to 0$.

Now suppose that the vector functions $\psi_i=\psi_i(t), i=1,2$ are solutions of the problems (conjugate system)

$$\psi_1(t-1)=\frac{\partial H_1[t,x_1,\psi_1]}{\partial x_1},t\in T_1, \quad (19)$$

$$\psi_1(t_1-1)=-\frac{\partial\varphi_1\big(x_1(t_1)\big)}{\partial x_1}+\frac{\partial M\big(\psi_2(t_1),x_1(t_1)\big)}{\partial x_1}, \quad (20)$$

$$\dot\psi_2(t)=-\frac{\partial H_2[t,x_2,\psi_2]}{\partial x_2},t\in T_2, \quad (21)$$

$$\psi_2(t_2)=-\frac{\partial\varphi_2\big(x_2(t_2)\big)}{\partial x_2}. \quad (22)$$

Then the formula of increment (17) will take the form

$$\Delta J(u_1,u_2)=-\sum_{t=t_0}^{t_1-1}\Delta_{\bar u_1}H_1[t,x_1,\psi_1]-\sum_{t=t_0}^{t_1-1}\frac{\partial\Delta_{\bar u_1}H_1'[t,x_1,\psi_1]}{\partial x_1}\Delta x_1(t)+o_1(\|\Delta x_1(t_1)\|)-$$
$$-\sum_{t=t_0}^{t_1-1}o_3(\|\Delta x_1(t)\|)-\int_{t_1}^{t_2}\Delta_{\bar u_2}H_2[t,x_2,\psi_2]dt-\int_{t_1}^{t_2}\frac{\partial\Delta_{\bar u_2}H_2{}'[t,x_2,\psi_2]}{\partial x_2}\Delta x_2(t)dt-$$
$$-\int_{t_1}^{t_2}o_5(\|\Delta x_2(t)\|)dt+o_2(\|\Delta x_2(t_2)\|)-o_4(\|\Delta x_1(t_1)\|). \quad (23)$$

In the following we will need to estimate the norm of the increment of the trajectory $\left(x_1(t), x_2(t)\right)$.

4. Estimating the norm of the increment $\Delta x_i(t), i = 1, 2$, of the trajectory $x_i(t), i = 1, 2$

Taking into account problem (8)-(9), and then applying the discrete analogue of the Fubini theorem (see, e.g., [9]), we obtain that

$$\Delta x_1(t+1) = \sum_{\tau=t_0}^{t}\left[f_1\left(\tau, \bar{x}_1(\tau), \bar{u}_1(\tau)\right) - f_1\left(\tau, x_1(\tau), u_1(\tau)\right)\right] +$$

$$+\sum_{\tau=t_0}^{t}\left[\sum_{s=\tau}^{t}\left[K_1\left(s, \tau, \bar{x}_1(\tau), \bar{u}_1(\tau)\right) - K_1\left(s, \tau, x_1(\tau), u_1(\tau)\right)\right]\right] - \sum_{\tau=t_0}^{t} \Delta x_1(\tau) =$$

$$= \sum_{\tau=t_0}^{t}\left[f_1\left(\tau, \bar{x}_1(\tau), \bar{u}_1(\tau)\right) - f_1\left(\tau, x_1(\tau), \bar{u}_1(\tau)\right) + \Delta_{\bar{u}_1} f_1[\tau]\right] +$$

$$+\sum_{\tau=t_0}^{t}\left[\sum_{s=\tau}^{t}\left[K_1\left(s, \tau, \bar{x}_1(\tau), \bar{u}_1(\tau)\right) - K_1\left(s, \tau, x_1(\tau), \bar{u}_1(\tau)\right)\right] + \Delta_{\bar{u}_1} K_1[s, \tau]\right] - \sum_{\tau=t_0}^{t} \Delta x_1(\tau).$$

Hence using the Lipschitz condition we obtain

$$\|\Delta x_1(t+1)\| \leq L_1\left[\sum_{\tau=t_0}^{t}\|\Delta x_1(\tau)\| + \sum_{\tau=t_0}^{t}\left[\left\|\Delta_{\bar{u}_1} f_1[\tau]\right\| + \sum_{s=\tau}^{t}\left\|\Delta_{\bar{u}_1} K_1[s, \tau]\right\|\right]\right].$$

($L_1 = const > 0$ is some constant)

Applying to the last inequality a discrete analogue of the Gronwall lemma (see, e.g., [6]) we obtain that

$$\|\Delta x_1(t)\| \leq L_2 \sum_{\tau=t_0}^{t}\left[\left\|\Delta_{\bar{u}_1} f_1[\tau]\right\| + \sum_{s=\tau}^{t}\left\|\Delta_{\bar{u}_1} F_1[s, \tau]\right\|\right], t \in T_2 \cup t_1. \tag{24}$$

$(L_2 = const > 0)$

Further proceeding from the system of equations (10) to the equivalent integral equation, we will have

$$\Delta x_2(t) = G\left(\bar{x}_1(t_1)\right) - G\left(x_1(t_1)\right) +$$

$$+\int_{t_1}^{t}\left[\int_{\tau}^{t}\left[K_2\left(s, \tau, \bar{x}_2(\tau), \bar{u}_2(\tau)\right) - K_2\left(s, \tau, x_2(\tau), \bar{u}_2(\tau)\right) + \Delta_{\bar{u}_2} K_2[s, \tau]\right] ds\right] d\tau +$$

$$+\int_{t_1}^{t}\left[\Delta_{\bar{u}_2} f_2[\tau] + f_2\left(\tau, \bar{x}_2(\tau), \bar{u}_2(\tau)\right) - f_2\left(\tau, x_2(\tau), \bar{u}_2(\tau)\right)\right] d\tau.$$

From the last identity, proceeding to the norm and using the Lipschitz condition, we obtain

$$\|\Delta x_2(t)\| \le L_3 \left[\|\Delta x_1(t_1)\| + \int_{t_1}^{t} \left[\int_{\tau}^{t} \left[\|\Delta x_2(\tau)\| + \left\|\Delta_{\bar{u}_2} K_2[s,\tau]\right\| \right] ds \right] d\tau + \right.$$
$$\left. + \int_{t_1}^{t} \left[\left\|\Delta_{\bar{u}_2} f_2[\tau]\right\| + \|\Delta x_2(\tau)\| \right] d\tau \right].$$
$$(L_3 = const > 0)$$

Applying the Gronwall-Bellman lemma to the last inequality, we arrive at the estimate

$$\|\Delta x_2(t)\| \le L_4 \left[\|\Delta x_1(t_1)\| + \int_{t_1}^{t} \left[\left\|\Delta_{\bar{u}_2} f_2[\tau]\right\| + \int_{\tau}^{t} \left\|\Delta_{\bar{u}_2} K_2[s,\tau]\right\| ds \right] d\tau. \right. \quad (25)$$
$$(L_4 = const > 0)$$

5. Necessary optimality conditions

Let us introduce the sets

$$f_1(t, x_1(t), U_1) = \{\alpha : \alpha = f_1(t, x_1(t), v_1(t)), v_1(t) \in U_1, t \in T_1\}, \quad (26)$$

$$F_1(t, \tau, x_1(\tau), U_1) = \{\beta : \beta = F_1(t, \tau, x_1(\tau), v_1(\tau)), v_1(\tau) \in U_1, \tau \in T_1\}, \quad (27)$$

$$K_1(t, \tau, x_1(\tau), U_1) = \{\gamma : \gamma = F_1(t, \tau, x_1(\tau), v_1(\tau)), v_1(\tau) \in U_1, \tau \in T_1\}. \quad (28)$$

Suppose that sets (26)-(28) are convex.

Suppose that in the problem under consideration $\Delta u_2(t) \equiv 0$.

The assumption of convexity of sets (26)-(28) allows a special increment of the control function $u_1(t)$ be determined from the formula

$$\Delta u_1(t;\varepsilon) = u_1(t;\varepsilon) - u_1(t), t \in T_1. \quad (29)$$

Here $\varepsilon \in [0,1]$ is an arbitrary number, and $u_1(t;\varepsilon)$ is such an arbitrary admissible control that for the corresponding arbitrary admissible control $v_1(t)$

$$f_1(t, x_1(t), u_1(t;\varepsilon)) - f_1(t, x_1(t), u_1(t)) = \varepsilon[f_1(t, x_1(t), v_1(t)) - f_1(t, x_1(t), u_1(t))], \quad (30)$$

$$K_1(t, \tau, x_1(\tau), u_1(\tau;\varepsilon)) - K_1(t, \tau, x_1(\tau), u_1(\tau)) =$$
$$= \varepsilon[K_1(t, \tau, x_1(\tau), v_1(\tau)) - K_1(t, \tau, x_1(\tau), u_1(\tau))], \quad (31)$$

$$F_1(t, \tau, x_1(\tau), u_1(\tau;\varepsilon)) - F_1(t, \tau, x_1(\tau), u_1(\tau)) =$$
$$= \varepsilon[F_1(t, \tau, x_1(\tau), v_1(\tau)) - F_1(t, \tau, x_1(\tau), u_1(\tau))]. \quad (32)$$

We denote by $(\Delta x_1(t;\varepsilon), \Delta x_2(t;\varepsilon))$ the special increment of the trajectory $(x_1(t), x_2(t))$ corresponding to special increment (29) of the control $u_1(t)$.

From estimates (24) and (25) it immediately follows that

$$\|\Delta x_1(t;\varepsilon)\| \le L_5\varepsilon, t \in T_1 t \in T_1 \cup t_1,$$
$$\|\Delta x_2(t;\varepsilon)\| = L_6\varepsilon, t \in T_2. \quad (33)$$

Now suppose that $\Delta u_1(t) = 0$. Then it is clear that $\Delta x_1(t) = 0$.

Then the special increment of the control function $u_2(t)$ can be determined from the formula

$$\Delta u_2(t;\mu)=\begin{cases}v_2-u_2(t), t\in[\theta,\theta+\mu),\\ 0, \qquad t\in T_2\setminus[\theta,\theta+\mu).\end{cases} \tag{34}$$

Here $\theta\in[t_1,t_2)$ is an arbitrary point of discontinuity of the control function $u_2(t)$, and $\mu>0$ is an arbitrary sufficiently small number such that $\theta+\mu<t_2$.

We denote by $\Delta x_2(t;\mu)$ the special increment of the trajectory $x_2(t)$.

Then from estimate (25) we obtain that

$$\|\Delta x_2(t;\mu)\|=L_7\mu, t\in T_2. \tag{35}$$

Taking into account formula (29) and estimates (33), we obtain from increment formula (23) that

$$J\big(u_1(t)+\Delta u_1(t;\varepsilon),u_2(t)\big)-J\big(u_1(t),u_2(t)\big)=$$
$$=-\varepsilon\sum_{t=t_0}^{t_1-1}\big[H_1\big(t,x_1(t),v_1(t),\psi_1(t)\big)-H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)\big]+o(\varepsilon). \tag{36}$$

Further, taking into account formula (34) and estimate (35) we have

$$J\big(u_1(t),u_2(t)+\Delta u_2(t;\mu)\big)-J\big(u_1(t),u_2(t)\big)=$$
$$=-\mu\int_{t_1}^{t_2}\big[H_2\big(t,x_2(t),v_2(t),\psi_2(t)\big)-H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)\big]\,dt+o(\mu). \tag{37}$$

From expansions (36) and (37) by virtue of arbitrariness of $\varepsilon\in[0,1]$ and $\mu>0$ follows

Theorem 1. Under the assumptions made, the optimality of the admissible control $\big(u_1(t),u_2(t)\big)$ requires that the inequalities

$$\Delta_{v_1}H_1[\theta,x_1(\theta)\psi_1(\theta)]\le 0,$$
$$\Delta_{v_2}H_2[\theta,x_2(\theta)\psi_2(\theta)]\le 0$$

hold for all $v_1\in U_1,\theta\in T_1$ and $v_2\in U_2$, $\theta\in[t_1,t_2)$, respectively.

The theorem is a necessary first-order optimality condition.

Since the optimal control problem under consideration is discrete-continuous, we will call the obtained optimality condition a discrete-continuous analogue of the Pontryagin maximum principle.

6. Conclusion

We consider a stepwise optimal control problem described by a set of Volterra difference and integro-differential equations.

Applying a modified version of the method of increments, taking into account the specifics of the problem under consideration, the global necessary first-order optimality condition is proved.

As was noted, the obtained result is a L.S. Pontryagin maximum principle-type necessary optimality condition. But maximum principle-type optimality conditions often degenerate (see, e.g., [10, 11]). Therefore, the author proposes to devote a separate article to a study of the case of degeneration of a discrete-continuous analogue of the Pontryagin maximum principle.

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