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On one control problem with variable structure

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ABSTRACT

A problem of optimal control with variable structure described in two domains by different linear Goursat-Darboux systems is considered. Necessary first- and second-order optimality conditions are established.

1. Introduction

In the works of A.I. Yegorov, V.I. Plotnikov and V.I. Sumin, S.S. Akhiyev and K.T. Akhmedov, T.K. Melikov, K.B. Mansimov, K.K. Hasanov and others, various optimal control problems described by systems of second-order hyperbolic equations with Goursat boundary conditions have been studied. A review of the relevant works is available, for example, in [1-6].

This article considers one optimal control problem, with variable structure, described in two domains by different Goursat-Darboux systems.

An analogue of L.S. Pontryagin's maximum principle is proved and the case of its degeneration is investigated.

Such optimal control problems are called step-wise optimal control problems or control problems with variable structure (see, e.g., [7-9]).

It should be noted that Goursat-Darboux boundary-value problems are often used for modeling sorption, desorption, drying, and chemical processes in chemical reactors (see, e.g., [1-6]).

2. Some notations and problem statement

Suppose that $D_i = [t_{i-1}, t_i] \times [x_0, x_1], i = 1,2$ are prescribed rectangles, $(T_i,\ X_i),\ i = \overline{1,k}$ $(t_0 < T_1 < T_2 < \cdots T_k \le t_1;\ x_0 < X_1 < X_2 < \cdots < X_k \le x_1)$. $\theta_i, i = \overline{1,k}$ $(t_1 < \theta_1 < \theta_2 < \cdots \theta_k \le t_2)$ are prescribed points, $U_1 \in R^r$ and $U_2 \in R^q$ are prescribed non-empty and bounded sets.

We denote by $u_1(t,x)$ and $u_2(t,x)$ measurable and bounded, respectively, r and -dimensional vector functions satisfying the constraints

$$u_1(t,x) \in U_1 \in R^r, \qquad (t,x) \in D_1 ,$$

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$$u_2(t,x) \in U_2 \in R^q, \qquad (t,x) \in D_2. \tag{1}$$

The pair $\left(u_1(t,x),\ u_2(t,x)\right)$ with the above properties will be called an admissible control.

Suppose that the controlled two-stage process in the domains of $D_i, i = 1,2$ is described by boundary-value problems

$$\frac{\partial^2 z_1}{\partial t \partial x} = A_1(t,x) z_1 + B_1(t,x)\frac{\partial z_1}{\partial t} + C_1(t,x)\frac{\partial z_1}{\partial x} + f_1(t,x,u_1), (t,x) \epsilon D_1, \tag{2}$$

$$z_1(t_0,x) = a(x),\ \ x\epsilon[x_0,x_1],\ \ z_1(t,x_0) = b_1(t),\ \ t\epsilon[t_0,t_1], \tag{3}$$

$$\frac{\partial^2 z_2}{\partial t \partial x} = A_2(t,x) z_2 + B_2(t,x)\frac{\partial z_2}{\partial t} + C_2(t,x)\frac{\partial z_2}{\partial x} + f_2(t,x,u_2), (t,x) \epsilon D_2, \tag{4}$$

$$z_2(t_1,x) = B(x) z_1(t_1,x),\ \ x\epsilon[x_0,x_1],$$
$$z_2(t,x_0) = b_2(t),\ \ t\epsilon[t_1,t_2], \quad B(x_0) z_1(t_1,x_0) = b_2(t_1). \tag{5}$$

Here, $A_i(t,x),\ B_i(t,x),\ C_i(t,x),\ i = 1,2$ are prescribed $(n \times n)$ measurable and bounded matrix functions, $f_i(t,x,u_i),\ i = 1,2$ are prescribed, -dimensional vector functions continuous over a set of variables, $B(x)$ is a prescribed continuous-differentiable n-dimensional vector function, $a(x), b_i(t), i = 1,2$ are prescribed absolutely continuous n-dimensional vector functions.

It is assumed that for each admissible control $\left(u_1(t,x),\ u_2(t,x)\right)$ there is the unique absolutely continuous solution $\left(z_1(t,x),\ z_2(t,x)\right)$ of boundary-value problem (2)-(5).

On the solutions of boundary-value problem (2)-(5) generated by all possible admissible controls, we define the functional

$$S(u_1,u_2) = \varphi_1\left(z_1(T_1,X_1),\cdots,z_1(T_k,X_k)\right) + \varphi_2\left(z_2(\theta_1,X_1),\cdots,z_2(\theta_k,X_k)\right) \tag{6}$$

and consider the problem of finding the minimum value of functional (6) under constraints (1)-(5).

Here $\varphi_1(c_1,c_2,\cdots,c_k)$ and $\varphi_2(d_1,d_2,\cdots,d_k)$ are prescribed, twice continuous-differentiable scalar functions.

The admissible control that affords the minimum value to functional (6) under constraints (1)-(5) will be called optimal control.

3. Formula of the second-order increment of the functional

Suppose that $\left(u_1(t,x),\ u_2(t,x), z_1(t,x),\ z_2(t,x)\ \right)$ and $\left(\bar{u}_1(t,x) = u_1(t,x) + \Delta u_1(t,x),\right.$ $\left.\bar{u}_2(t,x) = u_2(t,x) + \Delta u_2(t,x), \bar{z}_1(t,x) = z_1(t,x) + \Delta z_1(t,x), \bar{z}_2(t,x) = z_2(t,x) + \Delta z_2(t,x)\right) -$ are two admissible processes.

Then it is clear that $(\Delta z_1(t,x),\ \Delta z_2(t,x))$ is the solution of the boundary-value problem

$$\frac{\partial^2 \Delta z_1(t,x)}{\partial t \partial x} = A_1(t,x)\Delta z_1(t,x) + B_1(t,x)\frac{\partial \Delta z_1(t,x)}{\partial t} + C_1(t,x)\frac{\partial \Delta z_1(t,x)}{\partial x} +$$
$$+ f_1\left(t,x,\bar{u}_1(t,x)\right) - f_1\left(t,x,u_1(t,x)\right), (t,x) \in D_1, \tag{7}$$

$$\Delta z_1(t_0,x) = 0,\ \ x \in [x_0,x_1],$$
$$\Delta z_1(t,x_0) = 0,\ \ t \in [t_0,t_1], \tag{8}$$

$$\frac{\partial^2 \Delta z_2(t,x)}{\partial t \partial x} = A_2(t,x)\Delta z_2(t,x) + B_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial t} + C_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial x} +$$
$$+ f_2\left(t,x,\bar{u}_2(t,x)\right) - f_2\left(t,x,u_2(t,x)\right), (t,x) \in D_2, \tag{9}$$

$$\Delta z_2(t_1,x) = B(x)\Delta z_1(t_1,x),\ \ x \in [x_0,x_1],$$
$$\Delta z_1(t,x_0) = 0,\ \ t \in [t_1,t_2]. \tag{10}$$

Let us introduce the notations

$$\Delta_{\bar{u}_i} f_i[t,x] \equiv f_i\big(t,x,\bar{u}_i(t,x)\big) - H\big(t,x,u_i(t,x)\big),$$

$$H_i(t,x,u_i,\psi_i) = \psi_i{}' f_i(t,x,u_i),$$

$$\Delta_{\bar{u}_i} H[t,x,\psi_i] \equiv H_i\big(t,x,\bar{u}_i(t,x),\psi_i(t,x)\big) - H\big(t,x,u_i(t,x),\psi_i(t,x)\big), i = 1,2.$$

Here, $\psi_i(t,x)$, $i = 1,2$ are as yet unknown -dimensional vector functions, dash ($'$) indicates transpose.

Taking into account the introduced notations and applying the Taylor formula, the increment of quality functional (6) is written in the following form

$$\Delta S(u_1,u_2) = S(\bar{u}_1,\bar{u}_2) - S(u_1,u_2) = \sum_{i=1}^{k} \frac{\partial \varphi_1'\big(z_1(T_1,X_1),\cdots,z_1(T_k,X_k)\big)}{\partial c_i} \Delta z_1(T_i,X_i) +$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta z_1'(T_i,X_i)\frac{\partial^2 \varphi_1\big(z_1(T_1,X_1),\cdots,z_1(T_k,X_k)\big)}{\partial c_i \partial c_j}\Delta z_1\big(T_j,X_j\big) + o_1\left(\left[\sum_{i=1}^{k}\|\Delta z_1(T_i,X_i)\|\right]^2\right) +$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1} \psi_1'(t,x)\frac{\partial^2 \Delta z_1(t,x)}{\partial t \partial x}dxdt - \int_{t_0}^{t_1}\int_{x_0}^{x_1} \psi_1'(t,x)A_1(t,x)dxdt\, \Delta z_1(t,x) -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \psi_1'(t,x)B_1(t,x)\frac{\partial \Delta z_1(t,x)}{\partial t}dxdt - \int_{t_0}^{t_1}\int_{x_0}^{x_1} \psi_1'(t,x)C_1(t,x)\frac{\partial \Delta z_1(t,x)}{\partial x}dxdt -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\bar{u}_1(t,x)} H_1[t,x,\psi_1]dxdt + \sum_{i=1}^{k}\frac{\partial \varphi_2'\big(z_2(\theta_1,X_1),\ldots,z_2(\theta_k,X_k)\big)}{\partial d_i}\Delta\, z_2(\theta_i,X_i) +$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta z_2'(\theta_i,X_i)\frac{\partial^2 \varphi_2\big(z_2(\theta_1,X_1),\ldots,z_2(\theta_k,X_k)\big)}{\partial d_i \partial d_j}\Delta\, z_2\big(\theta_j,X_j\big) +$$

$$+o_2\left(\left[\sum_{i=1}^{k}\|\Delta z_2(\theta_i,X_i)\|\right]^2\right) + \int_{t_1}^{t_2}\int_{x_0}^{x_1} \psi_2'(t,x)\frac{\partial^2 \Delta z_2(t,x)}{\partial t \partial x}dxdt -$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1} \psi_2'(t,x)A_2(t,x)\frac{\partial^2 \Delta z_2(t,x)}{\partial t \partial x}dxdt - \int_{t_1}^{t_2}\int_{x_0}^{x_1} \psi_2'(t,x)B_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial t}dxdt -$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1} \psi_2'(t,x)C_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial x}dxdt - \int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{\bar{u}_2} H_2[t,x,\psi_2]dxdt. \tag{11}$$

Taking into account boundary conditions (8) and (10) we can verify the validity of the identities

$$\Delta z_1(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} \frac{\partial^2 \Delta z_1(\tau,s)}{\partial \tau \partial s}dsd\tau, \tag{12}$$

$$\frac{\partial \Delta z_1(t,x)}{\partial t} = \int_{x_0}^{x} \frac{\partial \Delta z_1(t,s)}{\partial s} ds, \tag{13}$$

$$\frac{\partial \Delta z_1(t,x)}{\partial x} = \int_{t_0}^{t} \frac{\partial \Delta z_1(\tau,x)}{\partial \tau} d\tau, \tag{14}$$

$$\Delta z_2(t,x) = \int_{t_0}^{t_1} \int_{x_0}^{x} B(x) \frac{\partial^2 \Delta z_1(\tau,s)}{\partial \tau \partial s} ds d\tau + \int_{t_1}^{t} \int_{x_0}^{x} \frac{\partial^2 \Delta z_2(\tau,s)}{\partial \tau \partial s} ds d\tau, \tag{15}$$

$$\frac{\partial \Delta z_2(t,x)}{\partial t} = \int_{x_0}^{x} \frac{\partial \Delta z_2(t,s)}{\partial s} ds d\tau, \tag{16}$$

$$\frac{\partial \Delta z_2(t,x)}{\partial x} = \int_{t_0}^{t_1} B(x) \frac{\partial^2 \Delta z_1(\tau,x)}{\partial \tau \partial s} d\tau + \int_{t_0}^{t_1} \int_{x_0}^{x} \frac{\partial B(x)}{\partial x} \frac{\partial^2 \Delta z_1(\tau,s)}{\partial \tau \partial s} ds d\tau +$$

$$+ \int_{t_1}^{t} \frac{\partial \Delta z_2(\tau,x)}{\partial \tau} d\tau. \tag{17}$$

Taking into account identities (12) and (15) it is proved that

$$\Delta z_1(T_i, X_i) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \alpha_i(t,x) \frac{\partial^2 \Delta z_1(t,x)}{\partial t \partial x} dx dt, \tag{18}$$

$$\Delta z_2(\theta_i, X_i) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \gamma_i(x) B(X_i) \frac{\partial^2 \Delta z_1(t,x)}{\partial t \partial x} dx dt$$

$$+ \int_{t_1}^{t_2} \int_{x_0}^{x_1} \beta_i(t,x) \frac{\partial^2 \Delta z_2(t,x)}{\partial t \partial x} dx dt. \tag{20}$$

Here $\alpha_i(t,x), \beta_i(t,x)$ are characteristic functions of the regions $[t_0, T_i] \times [x_0, X_i]$ and $[t_1, \theta_i] \times [x_0, X_i]$, and $\gamma_i(x)$ is a characteristic function of the segment $[x_0, X_i]$.

Given identities (12)-(20), we now assume that the vector functions $\psi_i(t,x)$, $i = 1,2$ are solutions, respectively, of the following integral equations

$$\psi_1(t,x) = -\sum_{i=1}^{k} \alpha_i(t,x) \frac{\partial \varphi_1\big(z_1(T_1,X_1), \cdots, z_1(T_k,X_k)\big)}{\partial c_i} +$$

$$+ \int_{t}^{t_1} \int_{x}^{x_1} A_1'(\tau,s)\psi_1(\tau,s) ds d\tau + \int_{x}^{x_1} B_1'(\tau,s)\psi_1(\tau,s) ds + \int_{t}^{t_1} C_1'(\tau,x)\psi_1(\tau,x) d\tau +$$

$$+ \int_{t_1}^{t_2} \int_{x}^{x_1} B'(s) A_2'(\tau,s)\psi_2(\tau,s) ds d\tau + \sum_{i=1}^{k} \gamma_i(x) B'(X_i) \frac{\partial \varphi_2\big(z_2(\theta_1,X_1), \ldots, z_2(\theta_k,X_k)\big)}{\partial d_i} +$$

$$+ \int_{t}^{t_2} B'(x) C_2'(\tau,x)\psi_2(\tau,x) d\tau + \int_{t_1}^{t_2} \int_{x}^{x_1} \frac{\partial B'(s)}{\partial s} C_2'(\tau,s)\psi_2(\tau,s) ds d\tau,$$

$$\psi_2(t,x) = -\sum_{i=1}^{k} \beta_i(t,x) \frac{\partial \varphi_2\big(z_2(\theta_1, X_1), \dots, z_2(\theta_k, X_k)\big)}{\partial d_i} + \int_t^{t_1} \int_x^{x_1} A_2'(\tau,s)\psi_2(\tau,s)dsd\tau +$$

$$-\int_x^{x_1} B_2'(t,s)\psi_2(t,s)ds + \int_t^{t_2} C_2'(\tau,x)\psi_2(\tau,x)d\tau.$$

Then increment formula (11) of quality functional (6) can be represented in the form

$$\Delta S(u_1,u_2) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta z_1'(T_i,X_i) \frac{\partial^2 \varphi_1\big(z_1(T_1,X_1),\cdots,z_1(T_k,X_k)\big)}{\partial c_i \partial c_j} \Delta z_1\big(T_j,X_j\big) +$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta z_2'(\theta_i,X_i) \frac{\partial^2 \varphi_2\big(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\big)}{\partial d_i \partial d_j} \Delta\, z_2\big(\theta_j,X_j\big) -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\bar{u}(t,x)} H_1[t,x,\psi_1]dxdt - \int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{\bar{u}_2} H_2[t,x,\psi_2]dxdt +$$

$$+o_1\left(\left[\sum_{i=1}^{k}\|\Delta z_1(T_i,X_i)\|\right]^2\right) + o_2\left(\left[\sum_{i=1}^{k}\|\Delta z_2(\theta_i,X_i)\|\right]^2\right). \quad (21)$$

From increment formula (21) we obtain that

$$S(\bar{u}_1,u_2) - S(u_1,u_2) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta z_1'(T_i,X_i) \frac{\partial^2 \varphi_1\big(z_1(T_1,X_1),\cdots,z_1(T_k,X_k)\big)}{\partial c_i \partial c_j} \Delta z_1\big(T_j,X_j\big) +$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta z_2'(\theta_i,X_i) \frac{\partial^2 \varphi_2\big(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\big)}{\partial d_i \partial d_j} \Delta\, z_2\big(\theta_j,X_j\big) -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\bar{u}_1(t,x)} H_1[t,x,\psi_1]dxdt + o_1\left(\left[\sum_{i=1}^{k}\|\Delta z_1(T_i,X_i)\|\right]^2\right), \quad (22)$$

$$S(u_1,\bar{u}_2) - S(u_1,u_2) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \Delta z_2'(\theta_i,X_i) \frac{\partial^2 \varphi_2\big(z_2(\theta_1,X_1),\dots,z_2(\theta_k,X_k)\big)}{\partial d_i \partial d_j} \Delta\, z_2\big(\theta_j,X_j\big) -$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{\bar{u}_2} H_2[t,x,\psi_2]dxdt + o_2\left(\left[\sum_{i=1}^{k}\|\Delta z_2(\theta_i,X_i)\|\right]^2\right). \quad (23)$$

Increment formulas (22) and (23) allow us in principle to obtain the necessary first-order optimality conditions.

In the future we will need formulas for solutions of boundary-value problems (7)-(8) and (9)-(10), both for estimating norms of state increments and for investigating the case of degeneration of L.S. Pontryagin's maximum principle.

Suppose $\Delta u_2(t,x) = 0$. Then we obtain that $\Delta z_1(t,x)$ is a solution of boundary-value problem (7)-(8), and $\Delta z_2(t,x)$ is a solution of the boundary-value problem

$$\frac{\partial^2 \Delta z_2(t,x)}{\partial t \partial x} = A_2(t,x)\Delta z_2(t,x) + B_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial t} + C_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial x}, \quad (24)$$

$$\Delta z_2(t_2,x) = B(x)\Delta z_1(t_1,x), \ \ x \in [x_0,x_1],$$
$$\Delta z_2(t,x_0) = 0, \ \ t \in [t_1,t_2]. \quad (25)$$

Let $(n \times n)$ matrix functions $R_i(t,x,\tau,s), i = 1,2$ be solutions of Volterra integral equations of the form

$$R_i(t,x,\tau,s) = E + \int_\tau^t \int_s^x R_i(t,x,\alpha,\beta)A_i(\alpha,\beta)d\alpha d\beta + \int_s^x R_i(t,x,\tau,\beta)B_i(\tau,\beta)d\beta +$$
$$+ \int_\tau^t R_i(t,x,\alpha,s)C_i(\alpha,s)d\alpha.$$

(E is an $(n \times n)$ identity matrix).

Following, for example, [10], the solutions of boundary-value problems (7)-(8) and (24)-(25) can be represented as

$$\Delta z_1(t,x) = \int_{t_0}^t \int_{x_0}^x R_1(t,x,\tau,s)\Delta_{\bar{u}(\tau,s)} f_1[\tau,s]d\tau ds, \quad (26)$$

$$\Delta z_2(t,x) = \int_{x_0}^x R_2(t,x,t_1,s)\left[\frac{\partial \Delta z_2(t_1,s)}{\partial s} - B_2(t_1,s)\Delta z_2(t_1,s)\right]ds. \quad (27)$$

Considering representation (26) from representation formula (27) after some transformations we obtain that

$$\Delta z_2(t,x) = \int_{t_0}^{t_1} \int_{x_0}^x R_2(t,x,t_1,s)\, Q_2(\tau,s)\Delta_{\bar{u}_2(\tau,s)} f_2[\tau,s]dsd\tau +$$
$$+ \int_{t_0}^{t_1} \int_{x_0}^x \int_{x_0}^s R_2(t,x,t_1,s)\, Q_1(s,\tau,\beta)\, \Delta_{\bar{u}_2(\tau,s)} f_2[\tau,\beta]dsd\tau d\beta. \quad (28)$$

Here, by definition

$$Q_1(s,\tau,\beta) = [B_s(s) - B_2(t_1,s)B(s)]R_1(t_1,s,\tau,\beta) + R(s)\frac{\partial R_1(t_1,s,\tau,\beta)}{\partial s},$$
$$Q_2(\tau,s) = B(s)R_1(t_1,s,\tau,s).$$

Then, applying the Fubini theorem (see, e.g., [11]) and introducing the notation

$$Q_3(t,x,\tau,s) = R_2(t,x,t_1,s)\, Q_2(\tau,s) + \int_s^x R_2(t,x,t_1,\beta)\, Q_1(\beta,\tau,s)d\beta,$$

from representation (28) we obtain that

$$\Delta z_2(t,x) = \int_{t_0}^{t_1} \int_{x_0}^x Q_3(t,x,\tau,s)\Delta_{\bar{u}_1(\tau,s)} f_1[\tau,s]dsd\tau. \quad (29)$$

Assuming that $\Delta u_1(t,x) = 0$ and $\Delta u_2(t,x) \neq 0$, it is clear that in this case $\Delta z_1(t,x) = 0$, and $\Delta z_2(t,x)$ is a solution of the boundary-value problem

$$\frac{\partial^2 \Delta z_2(t,x)}{\partial t \partial x} = A_2(t,x)\Delta z_2(t,x) + B_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial t} +$$
$$+ C_2(t,x)\frac{\partial \Delta z_2(t,x)}{\partial x} + \Delta_{\bar{u}_2(\tau,s)} f_2[\tau,s], \quad (30)$$

$$\Delta z_2(t_1,x)=0, x\in[x_0,x_1],$$
$$\Delta z_2(t,x_0)=0, t\in[t_1,t_2]. \tag{31}$$

Solution $\Delta z_2(t,x)$ of boundary-value problem (30)-(31) allows a representation in the form

$$\Delta z_2(t,x)=\int_{t_1}^{t}\int_{x_0}^{x}R_2(t,x,\tau,s)\Delta_{\overline{u}_2(\tau,s)}f_2[\tau,s]dsd\tau. \tag{32}$$

These representations (26) , (29), (32) allow us to obtain the necessary first- and second-order optimality conditions.

From representations (26) and (29) we obtain that

$$\|\Delta z_1(t,x)\|\le \mathcal{L}_1\int_{t_0}^{t}\int_{x_0}^{x}\left\|\Delta_{\overline{u}_1(\tau,s)}f_1[\tau,s]\right\|dsd\tau\,, (t,x)\epsilon D_1, \tag{33}$$

$$\|\Delta z_2(t,x)\|\le \mathcal{L}_2\int_{t_0}^{t_1}\int_{x_0}^{x}\left\|\Delta_{\overline{u}_1(\tau,s)}f_1[\tau,s]\right\|dsd\tau\,, (t,x)\epsilon D_2, \tag{34}$$

where $\mathcal{L}_i\ = const>0, i=1{,}2$ are some constants.

And from representation (32) it follows that

$$\|\Delta z_2(t,x)\|\le \mathcal{L}_3\int_{t_1}^{t}\int_{x_0}^{x}\Delta_{\overline{u}_2(\tau,s)}f_2[\tau,s]dsd\tau\,, (t,x)\epsilon D_2, \tag{35}$$

where $\mathcal{L}_3\ = const>0,$ is some constant.

Further, from representations (26) and (29) we obtain that

$$\Delta z_1(T_i,X_i)=\int_{t_0}^{t_1}\int_{x_0}^{x_1}\alpha_i(\tau,s)R_1(T_i,\ \ X_i,\tau,s)\Delta_{\overline{u}_1(\tau,s)}f_1[\tau,s]d\tau ds, \tag{36}$$

$$\Delta z_2(\theta_i,X_i)=\int_{t_0}^{t_1}\int_{x_0}^{x_1}\gamma_i(s)Q_3(\theta_i,X_i,\tau,s)\Delta_{\overline{u}_1(\tau,s)}f_1[\tau,s]dsd\tau. \tag{37}$$

Here $\alpha_i(t,x)$ is a characteristic function of the region $[t_0,T_i]\times[x_0,X_i]$, and $\gamma_i(x)$ is a characteristic function of the segment $[t_0,T_i]$.

And from representation (32) it follows that

$$\Delta z_2(\theta_i,X_i)=\int_{t_1}^{t_2}\int_{x_0}^{x_1}\beta_i(\tau,s)R_2(\theta_i,X_i,\tau,s)\Delta_{\overline{u}_2(\tau,s)}f_2[\tau,s]dsd\tau, \tag{38}$$

where $\beta_i(\tau,s)$ is a characteristic function of the region $[t_1,\theta_i]\times[x_0,X_i]$.

Taking into account formulas (36)-(38) we introduce notations of the form

$$K_1(\tau,s,\alpha,\beta)=$$
$$=-\sum_{i=1}^{k}\sum_{j=1}^{k}\Bigg(\gamma_i(s)\gamma_j(\beta)Q_3'(\theta_i,X_i,\tau,s)\frac{\partial^2\varphi_2\big(z_2(\theta_1,X_1),\cdots,z_2(\theta_k,X_k)\big)}{\partial d_i\partial d_j}Q_3\big(\theta_j,X_j,\alpha,\beta\big)+$$
$$+\alpha_i(\tau,s)\alpha_j(\alpha,\beta)R_1'(T_i,X_i,\tau,s)\frac{\partial^2\varphi_1\big(z_1(T_1,X_1),\cdots,z_1(T_k,X_k)\big)}{\partial c_i\partial c_j}R_1\big(T_j,X_j,\alpha,\beta\big)\Bigg), \tag{39}$$

$$K_2(\tau,s,\alpha,\beta)=-\sum_{i=1}^{k}\sum_{j=1}^{k}\beta_i(\tau,s)\beta_j(\alpha,\beta)R_2{}'(\theta_i,X_i,\tau,s)\times$$

$$\times \frac{\partial^2 \varphi_2\big(z_2(\theta_1, X_1), \cdots, z_2(\theta_k, X_k)\big)}{\partial d_i \partial d_j} R_2\big(\theta_j, X_j, \alpha, \beta\big). \tag{40}$$

Taking into account formulas (39) and (40), partial increments (22) and (23) of functional (6) are written in the form

$$S(\bar{u}_1, u_2) - S(u_1, u_2) = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\bar{u}_1(t,x)} H_1[t, x, \psi_1] dx dt +$$

$$+\frac{1}{2}\int_{D_1}\int_{D_1} \Delta_{\bar{u}_1(\tau,s)} f_1{}'[\tau, s] K_1(\tau, s, \alpha, \beta) \Delta_{\bar{u}_1(\alpha,\beta)} f_1[\alpha, \beta] ds d\tau d\alpha d\beta +$$

$$+o_1\left(\left[\sum_{i=1}^{k} \|\Delta z_1(T_i, X_i)\|\right]^2\right) + o_2\left(\left[\sum_{i=1}^{k} \|\Delta z_2(\theta_i, X_i)\|\right]^2\right), \tag{41}$$

$$S(u_1, \bar{u}_2) - S(u_1, u_2) = -\int_{t_1}^{t_2}\int_{x_0}^{x_1} \Delta_{\bar{u}_2} H_2[t, x, \psi_2] dx dt -$$

$$-\frac{1}{2}\int_{D_2}\int_{D_2} \Delta_{\bar{u}_2(\tau,s)} f_2'[\tau, s] K_2(\tau, s, \alpha, \beta) \Delta_{\bar{u}_2(\alpha,\beta)} f_2[\alpha, \beta] d\tau ds d\alpha d\beta + o_2\left(\left[\sum_{i=1}^{k} \|\Delta z_2(\theta_i, X_i)\|\right]^2\right). \tag{42}$$

4. Necessary optimality conditions

Suppose that $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ is an arbitrary Lebesgue point (see, e.g., [11, 12]) of the control $u_1(t, x)$, $v_1 \in U_1$is an arbitrary vector, and $\varepsilon > 0$ is an arbitrary small number such that $\theta + \varepsilon < t_1, \xi + \varepsilon < x_1$.

Special increment of the admissible control $u_1(t, x)$ will be determined from the formula

$$\Delta u_1(t, x; \varepsilon) = \begin{cases} v_1 - u_1(t, x), & (t, x) \in [\theta, \theta + \varepsilon) \times [\xi, \xi + \varepsilon), \\ 0, & (t, x) \overline{\in} [\theta, \theta + \varepsilon) \times [\xi, \xi + \varepsilon). \end{cases} \tag{43}$$

We denote by $\Delta z_1(t, x; \varepsilon)$ and $\Delta z_2(t, x; \varepsilon)$ special increments of states $z_1(t, x)$ and $z_2(t, x)$ corresponding to a special increment of the control $u_1(t, x)$.

From estimates (33), (34) it follows that

$$\|\Delta z_1(t, x; \varepsilon)\| \le N_1 \varepsilon^2,\ (t, x) \in D_1,$$
$$\|\Delta z_2(t, x; \varepsilon)\| \le N_2 \varepsilon^2,\ (t, x) \in D_2, \tag{44}$$

where $N_i = const > 0, i = 1{,}2$ are some constants.

Taking into account formulas (43) and (44) we obtain from increment formula (41) that

$$S(u_1(t, x) + \Delta u_1(t, x; \varepsilon), u_2(t, x)) - S\big(u_1(t, x), u_2(t, x)\big) =$$

$$= -\varepsilon^2 \Delta_{v_1} H_1[\theta, \xi, \psi_1] + \frac{\varepsilon^4}{2} \Delta_{v_1} f_1{}'[\theta, \xi] K_1(\theta, \xi, \theta, \xi) \Delta_{v_1} f_1[\theta, \xi] + o_3(\varepsilon^2) + o_4(\varepsilon^4). \tag{45}$$

Now we determine the special increment of the admissible control $u_2(t, x)$ from the formula

$$\Delta u_2(t, x; \mu) = \begin{cases} v_2 - u_2(t, x), & (t, x) \in [\theta, \theta + \mu) \times [\xi, \xi + \mu), \\ 0, & (t, x) \in D_2 \backslash [\theta, \theta + \mu) \times [\xi, \xi + \mu). \end{cases} \tag{46}$$

Here $(\theta, \xi) \in [t_1, t_2) \times [x_0, x_1)$ is an arbitrary Lebesgue point of the control $u_2(t, x)$, $v_2 \in U_2$

is an arbitrary vector, and $\mu > 0 -$ is an arbitrary small number such that $\theta + \mu < t_2, \xi + \mu < x_1$.

Let $\Delta z_2(t, x; \mu)$ be special increment of the state $z_2(t, x)$ corresponding to the special increment of the control $u_2(t, x)$.

From estimate (35) it follows that

$$\|\Delta z_2(t, x; \mu)\| \leq N_3 \mu^2,\ (t, x) \in D_2, \tag{47}$$

where $N_3 = const > 0,$ is some constant.

Considering formula (46) and estimate (47) from increment formula (42), we obtain that

$$S\big(u_1(t, x), u_2(t, x) + \Delta u_2(t, x; \mu)\big) - S\big(u_1(t, x), u_2(t, x)\big) =$$

$$= -\mu^2 \Delta_{v_2} H_2[\theta, \xi, \psi_2] + \frac{\mu^4}{2} \Delta_{v_2} {f_2}'[\theta, \xi] K_2(\theta, \xi, \theta, \xi) \Delta_{v_2} f_2[\theta, \xi] + o_5(\mu^2) + o_6(\mu^4). \tag{48}$$

From expansions (46) and (48) follows

Theorem 1. The optimality of the admissible control $\big(u_1(t, x), u_2(t, x)\big)$ requires that the relations

$$\max_{u_1 \in U_1} H_1\,(\theta, \xi, z_1(\theta, \xi), v_1, \psi_1(\theta, \xi)) = H_1(\theta, \xi, z_1(\theta, \xi), u_1(\theta, \xi), \psi_1(\theta, \xi)), \tag{49}$$

$$\max_{u_2 \in U_2} H_2\,(\theta, \xi, z_2(\theta, \xi), v_2, \psi_2(\theta, \xi)) = H_2(\theta, \xi, z_2(\theta, \xi), u_2(t, x), \psi_2(\theta, \xi)) \tag{50}$$

hold for all $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ and $(\theta, \xi) \in [t_1, t_2) \times [x_0, x_1)$, respectively.

Relations (49) and (50), being necessary first-order optimality conditions represent an analogue of the classical L.S. Pontryagin maximum principle for the problem under consideration (see, for example, [1-6]).

But in some cases, they can degenerate.

Such cases are called singular cases and the corresponding controls are called singular controls (see, e.g., [2, 6, 13]).

Definition. The admissible control $\big(u_1(t, x), u_2(t, x)\big)$ will be called singular, in the sense of the Pontryagin maximum principle, if the relations

$$H_1(\theta, \xi, z_1(\theta, \xi), v_1, \psi_1(\theta, \xi)) = H_1(\theta, \xi, z_1(\theta, \xi), u_1(\theta, \xi), \psi_1(\theta, \xi)),$$

$$H_2(\theta, \xi, z_2(\theta, \xi), v_2, \psi_2(\theta, \xi)) = H_2(\theta, \xi, z_2(\theta, \xi), u_2(t, x), \psi_2(\theta, \xi))$$

hold for all $v_1 \in U_1, (\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ and $v_2 \in U_2, (\theta, \xi) \in [t_1, t_2) \times [x_0, x_1)$, respectively.

By means of expansions (45) and (48) we prove

Theorem 2. The optimality of the control $\big(u_1(t, x), u_2(t, x)\big)$ singular in the sense of the Pontryagin maximum principle requires that the inequalities

$$\Delta_{v_1} {f_1}'[\theta, \xi] K_1(\theta, \xi, \theta, \xi) \Delta_{v_1} f_1[\theta, \xi] \leq 0,$$

$$\Delta_{v_2} {f_2}'[\theta, \xi] K_2(\theta, \xi, \theta, \xi) \Delta_{v_2} f_2[\theta, \xi] \leq 0$$

hold for all $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ and $(\theta, \xi) \in [t_1, t_2) \times [x_0, x_1)$, respectively.

5. Conclusion

The article considers one two-stage optimal control problem described by two boundary conditions for linear in state and nonlinear in control second-order hyperbolic equations and a nonlinear multipoint quality functional.

A formula for the second-order increment of the quality criterion that corresponds to two admissible controls and is constructive in nature is built.

Investigating the constructed increment formula, on the needle-type variations of control functions and at the same time, taking into account their independence, an analogue of the Pontryagin maximum principle is established and the case of its degeneration is investigated.

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