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Necessary first- and second-order optimality conditions in one two-stage stepwise control problem

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ABSTRACT

A discrete stepwise optimal control problem described by two systems of Volterra-type difference equations of general form is investigated. Under the assumption of openness of the control domains, the necessary first- and second-order optimality conditions are established.

1. Introduction

Stepwise optimal control problems described by ordinary differential equations or difference equations have been studied in [1-4] and others.

In the proposed study, we consider one stepwise optimal control problem described by general Volterra-type nonlinear difference equations.

Under the assumption of openness of control regions, the first and second variations of the quality functional are calculated and explicit necessary first- and second-order optimality conditions are proved.

2. Problem statement

Let the controlled two-stage process be described by systems of difference equations

$$x_i(t+1) = f_i\big(t, x_i(t), u_i(t)\big) + \\ + \sum_{\tau=t_{i-1}}^{t} K_i\big(t, \tau, x_i(\tau), u_i(\tau)\big), t \in T_i = \{t_{i-1}, t_{i-1}+1, \dots, t_i - 1\}, i = 1,2, \tag{1}$$

with initial conditions

$$x_1(t_0) = x_{10}, \tag{2}$$

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$$x_2(t_1) = G\big(x_1(t_1)\big).$$

Here, $f_i(t, x_i, u_i)$ $\big(K_i(t, \tau, x_i, u_i)\big), i = 1{,}2$ are prescribed -dimensional vector functions discrete in t (t, τ), and continuous in (x_i, u_i), together with partial derivatives with respect to (x_i, u_i) up to and including the second order, $u_1(t)$ $\big(u_2(t)\big) - r(q)$ is a n-dimensional discrete function of control actions, with values from the prescribed non-empty, bounded and open set U_1 (U_2), i.e.

$$u_1(t) \in U_1 \subset R^r, t \in T_1,$$
$$u_2(t) \in U_2 \subset R^q, t \in T_2, \tag{3}$$

x_{10} is a prescribed -dimensional constant vector, $G(x_1)$ is a prescribed n-dimensional vector function with a second continuous derivative.

The pair $(u_1(t), u_2(t))$ with the above properties will be called an admissible control.

Suppose that $\varphi_i(x_i), i = 1{,}2$ is twice prescribed continuously differentiable scalar functions, $F_i(t, \tau, x_i, u_i), i = 1{,}2$ is prescribed scalar functions, discrete in (t, τ) and twice continuously differentiable in (x_i, u_i) for all (t, τ).

Let us consider the problem of finding the minimum value of the Bolza functional

$$J(u_1, u_2) = \sum_{i=1}^{2} \varphi_i\big(x_i(t_i)\big) + \sum_{i=1}^{2} \left[\sum_{t=t_{i-1}}^{t_i - 1} \left[\sum_{\tau=t_{i-1}}^{t} F_i\big(t, \tau, x_i(\tau), u_i(\tau)\big) \right] \right] \tag{4}$$

under constraints (1)-(3).

The admissible control delivering the minimum value to functional (4) under constraints (1)-(3) will be called the optimal control, and the respective process $\big(u_1(t), u_2(t), x_1(t), x_2(t)\big)$ will be called the optimal process.

3. Formula for the increment of the quality functional and calculation of functional variations

Suppose that $\big(u_1(t), u_2(t), x_1(t), x_2(t)\big)$ and $\big(\bar{u}_1(t) = u_1(t) + \Delta u_1(t),\ \bar{u}_2(t) = u_2(t) + \Delta u_2(t), \bar{x}_1(t) = x_1(t) + \Delta x_1(t), \bar{x}_2(t) = x_2(t) + \Delta x_2(t)\big)$ are two admissible processes.

Then, it is clear that $\big(\Delta x_1(t), \Delta x_2(t)\big)$ will be the solution to the problem

$$\Delta x_i(t+1) = f_i\big(t, \bar{x}_i(t), \bar{u}_i(t)\big) - f_i\big(t, x_i(t), u_i(t)\big) +$$
$$+ \sum_{\tau=t_{i-1}}^{t} \left[K_i\big(t, \tau, \bar{x}_i(\tau), \bar{u}_i(\tau)\big) - K_i\big(t, \tau, x_i(\tau), u_i(\tau)\big)\right], \qquad i = 1{,}2, \tag{5}$$

$$\Delta x_1(t_0) = 0, t \in T_1,$$
$$\Delta x_2(t_1) = G\big(\bar{x}_1(t)\big) - G\big(x_1(t_1)\big). \tag{6}$$

Let us introduce analogs of the Hamilton-Pontryagin function in the form

$$H_i\big(t, x_i(t), u_i(t), \psi_i(t)\big) = \psi_i'(t) f_i\big(t, x_i(t), u_i(t)\big) +$$
$$+ \sum_{\tau=t}^{t_i - 1} \left[\psi_i'(\tau) K_i\big(\tau, t, x_i(t), u_i(t)\big) - F_i\big(\tau, t, x_i(t), u_i(t)\big)\right], \qquad i = 1{,}2, \tag{7}$$

$$M(\psi_2(t_1 - 1), x_1) = \psi_2{}'(t_1 - 1) G(x_1).$$

Here $\psi_i = \psi_i(t), i = 1{,}2$ are as yet arbitrary, n-dimensional vector functions (vector functions of adjoint variables), and the dash denotes the transpose operation.

Taking into account formulas (5), (6), (7) after some transformations, the formula for the increment of quality functional (4) takes the following form:

$$\Delta J(u_1,u_2)=\sum_{i=1}^{2}\left[\varphi_i\big(\bar{x}_i(t_i)\big)-\varphi_i\big(x_i(t_i)\big)\right]+\sum_{i=1}^{2}\left[\sum_{t=t_{i-1}}^{t_i-1}\psi_i'(t)\Delta x_i(t_i+1)\right]-$$

$$-\sum_{i=1}^{2}\left[H_i\big(t,\bar{x}_i(t),\bar{u}_i(t),\psi_i(t)\big)-H_i\big(t,x_i(t),u_i(t),\psi_i(t)\big)\right]. \qquad (8)$$

Taking into account initial conditions (2) it is proved that

$$\sum_{t=t_0}^{t_1-1}\psi_1{}'(t)\Delta x_1(t+1)=\psi_1'(t_1-1)\Delta x_1(t_1)+\sum_{t=t_0}^{t_1-1}\psi_1{}'(t-1)\Delta x_1(t), \qquad (9)$$

$$\sum_{t=t_1}^{t_2-1}\psi_2{}'(t)\Delta x_2(t+1)=\psi_2{}'(t_2-1)\Delta x_2(t_2)-$$

$$-\Big(M\big(\psi_2(t_1-1),\bar{x}_1(t_1)\big)-M\big(\psi_2(t_1-1),x_1(t_1)\big)\Big)+\sum_{t=t_1}^{t_2-1}\psi_2'(t-1)\Delta x_2(t). \qquad (10)$$

Taking into account identities (9) and (10), and using the Taylor formula, increment (8) of the quality functional is written in the following form

$$\Delta J(u_1,u_2)=\sum_{i=1}^{2}\frac{\partial\varphi_i'\big(x_i(t_i)\big)}{\partial x_i}\Delta x_i(t_i)+\frac{1}{2}\sum_{i=1}^{2}\Delta x_i'(t_i)\frac{\partial^2\varphi_i\big(x_i(t_i)\big)}{\partial x_i^2}\Delta x_i(t_i)+$$

$$+\sum_{i=1}^{2}o_i(\|\Delta x_i(t_i)\|^2)+\psi_1'(t_1-1)\Delta x_1(t_1)+\sum_{t=t_0}^{t_1-1}\psi_1{}'(t-1)\Delta x_1(t)-$$

$$-\sum_{t=t_0}^{t_1-1}\left[\frac{\partial H_1'\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial x_1}\Delta x_1(t)+\frac{\partial H_1'\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1}\Delta u_1(t)\right]-$$

$$-\frac{1}{2}\sum_{t=t_0}^{t_1-1}\left[\Delta x_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial x_1^2}\Delta x_1(t)\right.$$

$$+2\Delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1\partial x_1}\Delta x_1(t)+$$

$$\left.+\Delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1^2}\Delta u_1(t)\right]-\sum_{t=t_0}^{t_1-1}o_3([\|\Delta x_1(t)\|+\|\Delta u_1(t)\|]^2)+$$

$$+\sum_{t=t_1}^{t_2-1}\psi_2{}'(t-1)\Delta x_2(t)+\psi_2{}'(t_2-1)\Delta x_2(t_2)-$$

$$-\sum_{t=t_1}^{t_2-1}\left[\frac{\partial H_2'\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial x_2}\Delta x_2(t)+\frac{\partial H_2'\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2}\Delta u_2(t)\right]-$$

$$-\frac{1}{2}\sum_{t=t_1}^{t_2-1}\left[\Delta x_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial x_2^2}\Delta x_2(t)\right.$$

$$+2\Delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2\partial x_2}\Delta x_2(t)+$$

$$+\Delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2^2}\Delta u_2(t)\Bigg]-\sum_{t=t_1}^{t_2-1}o_4([\|\Delta x_2(t)\|+\|\Delta u_2(t)\|]^2)-$$
$$-\frac{\partial M'\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1}\Delta x_1(t_1)-\frac{1}{2}\Delta x_1'(t_1)\frac{\partial^2 M\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1^2}\Delta x_1(t_1)-$$
$$-o_5(\|\Delta x_1(t_1)\|^2). \quad (11)$$

Here, and in the following, $\|\alpha\|$ is the norm of the vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)'$, determined by the formula $\|\alpha\|=\sum_{i=1}^{n}|\alpha_i|$, and $o(\alpha^2)$ is a value with a higher order of smallness than α^2, i.e. $\frac{o(\alpha^2)}{\alpha^2}\to 0$ at $\alpha\to 0$.

Suppose that the vector functions $\psi_i=\psi_i(t), i=1,2$ are solutions of Cauchy problems in the form

$$\psi_i(t-1)=\frac{\partial H_i\big(t,x_i(t),u_i(t),\psi_i(t)\big)}{\partial x_i}, i=1,2,$$
$$\psi_1(t_1-1)=-\frac{\partial \varphi_i\big(x_i(t_i)\big)}{\partial x_i}+\frac{\partial M\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1},$$
$$\psi_2(t_2-1)=-\frac{\partial \varphi_2\big(x_2(t_2)\big)}{\partial x_2}.$$

Then formula of the increment (11) will take the form

$$\Delta J(u_1,u_2)=-\sum_{t=t_0}^{t_1-1}\frac{\partial H_1'\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1}\Delta u_1(t)+\frac{1}{2}\Delta x_1'(t_1)\frac{\partial^2\varphi_1\big(x_1(t_1)\big)}{\partial x_1^2}\Delta x_1(t_1)+$$
$$+\frac{1}{2}\Delta x_2'(t_2)\frac{\partial^2\varphi_2\big(x_2(t_2)\big)}{\partial x_2^2}\Delta x_2(t_2)-\frac{1}{2}\Delta x_1'(t_1)\frac{\partial^2 M\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1^2}\Delta x_1(t_1)-$$
$$-\frac{1}{2}\sum_{t=t_0}^{t_1-1}\Bigg[\Delta x_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial x_1^2}\Delta x_1(t)$$
$$+2\Delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1\partial x_1}\Delta x_1(t)+$$
$$+\Delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1^2}\Delta u_1(t)\Bigg]-$$
$$-\frac{1}{2}\sum_{t=t_1}^{t_2-1}\Bigg[\Delta x_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial x_2^2}\Delta x_2(t)$$
$$+2\Delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2\partial x_2}\Delta x_2(t)+$$
$$+\Delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2^2}\Delta u_2(t)\Bigg]+\sum_{i=1}^{2}o_i(\|\Delta x_i(t_i)\|^2)-$$
$$-\sum_{t=t_0}^{t_1-1}o_3([\|\Delta x_1(t)\|+\|\Delta u_1(t)\|]^2)-\sum_{t=t_1}^{t_2-1}o_4([\|\Delta x_2(t)\|+\|\Delta u_2(t)\|]^2)-o_5(\|\Delta x_1(t_1)\|^2). \quad (12)$$

4. Estimation of the rate of trajectory increment

To derive the necessary optimality conditions, using constructed increment formula (14) of functional (4), we need an estimate of the rate of trajectory increments.

To this end, let us consider two possible cases.

First suppose that $\Delta u_2(t) = 0$. Then from problems (5)-(6) we have that

$$\Delta x_1(t+1) = f_1\big(t, \bar{x}_1(t), u_1(t)\big) - f_2\big(t, x_1(t), u_1(t)\big) + \\ + \sum_{\tau=t_0}^{t} \big[K_1\big(t, \tau, \bar{x}_1(\tau), \bar{u}_1(\tau)\big) - K_1\big(t, \tau, x_1(\tau), u_1(\tau)\big)\big], \tag{13}$$

$$\Delta x_1(t_0) = 0, \tag{14}$$

$$\Delta x_2(t+1) = f_2\big(t, \bar{x}_2(t), u_2(t)\big) - f_2\big(t, x_2(t), u_2(t)\big) + \\ + \sum_{\tau=t_1}^{t} \big[K_2\big(t, \tau, \bar{x}_2(\tau), u_2(\tau)\big) - K_2\big(t, \tau, x_2(\tau), u_2(\tau)\big)\big], \tag{15}$$

$$\Delta x_2(t_1) = G\big(\bar{x}_1(t_1)\big) - G\big(x_1(t_1)\big). \tag{16}$$

Let ε be a number that is sufficiently small in absolute value, and $\delta u_1(t) \in R^r, t \in T_1$ is an arbitrary r-dimensional discrete and bounded vector function (a variation of the control function $u_1(t)$).

We denote by $\big(\Delta x_1(t;\varepsilon), \Delta x_2(t;\varepsilon)\big)$ the special increment of the trajectory $\big(x_1(t), x_2(t)\big)$, corresponding to the special increment

$$\Delta u_1(t;\varepsilon) = \varepsilon \delta u_1(t) \tag{17}$$

of the admissible control $\big(u_1(t), u_2(t)\big)$.

Using problems (13)-(14) and (15)-(16) and smoothness conditions imposed on the right-hand sides of these equations, we prove

Lemma 1. Under the assumptions made, the following expansions take place:

$$\Delta x_1(t;\varepsilon) = \varepsilon \delta x_1(t) + o(\varepsilon; t),$$

$$\Delta x_2(t;\varepsilon) = \varepsilon \delta x_2(t) + o(\varepsilon; t). \tag{18}$$

Here $\delta x_1(t)$ and $\delta x_2(t)$ (variations of trajectories (see, e.g., [5-7])) are solutions, respectively, of the following problems (equations in variations [6, 7]):

$$\delta x_1(t+1) = \frac{\partial f_1\big(t, x_1(t), u_1(t)\big)}{\partial x_1} \delta x_1(t) + \frac{\partial f_1\big(t, x_1(t), u_1(t)\big)}{\partial u_1} \delta u_1(t) + \\ + \sum_{\tau=t_0}^{t} \left[\frac{\partial K_1\big(t, \tau, x_1(\tau), u_1(\tau)\big)}{\partial x_1} \delta x_1(\tau) + \frac{\partial K_1\big(t, \tau, x_1(\tau), u_1(\tau)\big)}{\partial u_1} \delta u_1(\tau)\right], t \in T_1, \tag{19}$$

$$\delta x_1(t_0) = 0, \tag{20}$$

$$\delta x_2(t+1) = \frac{\partial f_2\big(t, x_2(t), u_2(t)\big)}{\partial x_2} \delta x_2(t) + \sum_{\tau=t_1}^{t} \frac{\partial K_2\big(t, \tau, x_2(\tau), u_2(\tau)\big)}{\partial x_2} \delta x_2(\tau), t \in T_2, \tag{21}$$

$$\delta x_2(t_1) = \frac{\partial G\big(x_1(t_1)\big)}{\partial x_1} \delta x_1(t_1). \tag{22}$$

Now, let $\Delta u_1(t) = 0$, $\Delta u_2(t) \neq 0, \mu$ be a number that is sufficiently small in absolute value, and $\delta u_2(t) \in R^q, t \in T_2$ is an arbitrary -dimensional discrete and bounded vector function.

Determine the special increment of the control function $u_2(t)$ by the formula

$$\Delta u_2(t;\mu) = \mu \delta u_2(t), \quad t \in T_2. \tag{23}$$

It is clear that here, $\Delta x_1(t;\mu)=0$.

Then from problem (5)-(6) at $i=2$ we obtain that at this $\Delta x_2(t;\mu)$ is the solution of the problem

$$\Delta x_2(t+1;\mu)=f_2\big(t,x_2(t)+\Delta x_2(t;\mu),u_2(t)+\Delta u_2(t;\mu)\big)-f_2\big(t,x_2(t),u_2(t)\big)+$$
$$+\sum_{\tau=t_1}^{t}\big[K_2\big(t,\tau,x_2(\tau)+\Delta x_2(\tau;\mu),u_2(\tau)+\Delta u_2(\tau;\mu)\big)-K_2\big(t,\tau,x_2(\tau),u_2(\tau)\big)\big],t\in T_2, \quad (24)$$
$$\Delta x_2(t_1;\mu)=0. \quad (25)$$

Taking into account problem (24)-(25), we prove

Lemma 2. Under the assumptions made, the following expansion takes place:

$$\Delta x_2(t;\mu)=\mu y(t)+o(\mu;t), \quad (26)$$

where $\delta y(t)$ is the solution to the problem

$$y(t+1)=\frac{\partial f_2\big(t,x_2(t),u_2(t)\big)}{\partial x_2}y(t)+\frac{\partial f_2\big(t,x_2(t),u_2(t)\big)}{\partial u_2}\delta u_2(t)+$$
$$+\sum_{\tau=t_1}^{t}\left[\frac{\partial K_2\big(t,\tau,x_2(\tau),u_2(\tau)\big)}{\partial x_2}y(\tau)+\frac{\partial K_2\big(t,\tau,x_2(\tau),u_2(\tau)\big)}{\partial u_2}\delta u_2(\tau)\right],t\in T_2, \quad (27)$$
$$y(t_1)=0. \quad (28)$$

5. Necessary optimality conditions

Using expansions (18), (26), (27), (28) and formulas (17), (23) from increment formula (12) we obtain the validity of the expansions

$$J\big(u_1(t)+\varepsilon\delta u_1(t),u_2(t)\big)-J\big(u_1(t),u_2(t)\big)=$$
$$=-\varepsilon\sum_{t=t_0}^{t_1-1}\frac{\partial H_1'\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1}\delta u_1(t)-$$
$$-\frac{\varepsilon^2}{2}\left[\sum_{t=t_0}^{t_1-1}\left[\delta x_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial x_1^2}\delta x_1(t)+\right.\right.$$
$$\left.\left.+2\delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1\partial x_1}\delta x_1(t)+\delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1^2}\delta u_1(t)\right]\right]+$$
$$+\frac{\varepsilon^2}{2}\delta x_1'(t_1)\frac{\partial^2\varphi_1\big(x_1(t_1)\big)}{\partial x_1^2}\delta x_1(t_1)+\frac{\varepsilon^2}{2}\delta x_2'(t_2)\frac{\partial^2\varphi_2\big(x_2(t_2)\big)}{\partial x_2^2}\delta x_2(t_2)-$$
$$-\frac{\varepsilon^2}{2}\delta x_1'(t_1)\frac{\partial^2 M\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1^2}\delta x_1(t_1)-$$
$$-\frac{\varepsilon^2}{2}\sum_{t=t_1}^{t_2-1}\delta x_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial x_2^2}\delta x_2(t)+o(\varepsilon^2). \quad (29)$$

$$J\big(u_1(t),u_2(t)+\mu\delta u_2(t)\big)-J\big(u_1(t),u_2(t)\big)=$$
$$=\frac{\mu^2}{2}y'(t_2)\frac{\partial^2\varphi_2\big(x_2(t_2)\big)}{\partial x_2^2}y(t_2)-\mu\sum_{t=t_1}^{t_2-1}\frac{\partial\, H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2}\delta u_2(t)+$$

$$-\frac{\mu^2}{2}\sum_{t=t_1}^{t_2-1}\left[y'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial x_2^2}y(t)+\right.$$
$$+2\delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2\partial x_2}y(t)+$$
$$\left.+\delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2^2}\delta u_2(t)\right]+o(\mu^2). \tag{30}$$

From expansions (29) and (30) it follows that the first and second variations of quality functional (4) have, respectively, the following form

$$\delta^1 J(u_1,u_2;\delta u_1)=-\sum_{t=t_0}^{t_1-1}\frac{\partial H_1'\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1}\delta u_1(t), \tag{31}$$

$$\delta^1 J(u_1,u_2;\delta u_2)=-\sum_{t=t_1}^{t_2-1}\frac{\partial\, H_2'\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2}\delta u_2(t), \tag{32}$$

$$\delta^2 J(u_1,u_2;\delta u_1)=\delta x_1'(t_1)\frac{\partial^2\varphi_1\big(x_1(t_1)\big)}{\partial x_1^2}\delta x_1(t_1)+\delta x_2'(t_2)\frac{\partial^2\varphi_2\big(x_2(t_2)\big)}{\partial x_2^2}\delta x_2(t_2)-$$
$$-\delta x_1'(t_1)\frac{\partial^2 M\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1^2}\delta x_1(t_1)-$$
$$-\sum_{t=t_0}^{t_1-1}\left[\delta x_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial x_1^2}\delta x_1(t)+\right.$$
$$+2\delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1\partial x_1}\delta x_1(t)+$$
$$\left.+\delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1^2}\delta u_1(t)\right]-$$
$$-\sum_{t=t_1}^{t_2-1}\delta x_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial x_2^2}\delta x_2(t), \tag{33}$$

$$\delta^2 J(u_1,u_2;\delta u_2)=y'(t_2)\frac{\partial^2\varphi_2\big(x_2(t_2)\big)}{\partial x_2^2}y(t_2)-$$
$$-\sum_{t=t_1}^{t_2-1}\left[y'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial x_2^2}y(t)+\right.$$
$$+2\delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2\partial x_2}y(t)+$$
$$\left.+\delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2^2}\delta u_2(t)\right]. \tag{34}$$

Taking into account the main result of the calculus of variations (see, e.g., [5, 6]), we obtain from formulas (31), (32), (33), (34) that optimality of the admissible control $\big(u_1(t),u_2(t)\big)$ in the considered problem requires that the relations

$$\delta^1 J(u_1,u_2;\delta u_1)=0, \tag{35}$$

$$\delta^2 J(u_1,u_2;\delta u_1)\geq 0, \tag{36}$$

$$\delta^1 J(u_1, u_2; \delta u_2) = 0, \tag{37}$$

$$\delta^2 J(u_1, u_2; \delta u_2) \geq 0, \tag{38}$$

hold for all δu_1 and δu_2, respectively.

Identities (35) and (37) are implicit necessary first-order optimality conditions, and inequalities (36) and (38) are implicit necessary second-order optimality conditions.

Using the arbitrariness of δu_1 and δu_2, from identities (35) and (37) we obtain the validity of the first-order optimality condition of the Euler equation analog type [6].

Theorem 1. The optimality of the admissible control $(u_1(t), u_2(t))$ requires that the relations

$$\frac{\partial H_1(\theta, x_1(\theta), u_1(\theta), \psi_1(\theta))}{\partial u_1} = 0, \tag{39}$$

$$\frac{\partial H_2(\theta, x_2(\theta), u_2(\theta), \psi_2(\theta))}{\partial u_2} = 0 \tag{40}$$

hold for all $\theta \in T_1$ and $\theta \in T_2$, respectively.

The analogs of Euler equations (39)-(40) are explicit necessary first-order optimality conditions.

Each admissible control $(u_1(t), u_2(t))$ satisfying relations (29), (40), will be called, following, for example, [6], a classical extremal.

It is known that (see, e.g., [6]) the number of classical extremals can be quite large.

A multitude set of classical extremals suspicious for optimality can narrow down the necessary second-order optimality conditions, which are explicit in nature.

Taking this into account, from inequalities (36) and (38) we obtain the explicit necessary optimality conditions.

Let $F_i(t,\tau)$ be (n×n) matrix functions that are solutions of the problems

$$F_1(t,\tau-1) = F_1(t,\tau)\frac{\partial f_1(t, x_1(t), u_1(t))}{\partial x_1} + \sum_{s=\tau}^{t-1} F_1(t,s)\frac{\partial K_1(s,\tau,x_1(\tau),u_1(\tau))}{\partial x_1},$$

$$F_1(t,t-1) = E,$$

$$F_2(t,\tau-1) = F_2(t,\tau)\frac{\partial f_2(t, x_2(t), u_2(t))}{\partial x_2} + \sum_{s=\tau}^{t-1} F_2(t,s)\frac{\partial K_2(s,\tau,x_2(\tau),u_2(\tau))}{\partial x_2},$$

$$F_2(t,t-1) = E,$$

where E is a unit matrix.

Then the solutions of problems (19)-(20), (21)-(22) can be represented respectively as

$$\delta x_1(t) = \sum_{\tau=t_0}^{t} \frac{\partial f_1(\tau, x_1(\tau), u_1(\tau))}{\partial u_1}\delta u_1(\tau) + \sum_{\tau=t_0}^{t}\left[\sum_{s=\tau}^{t-1} F_1(t,s)\frac{\partial K_1(s,\tau,x_1(\tau),u_1(\tau))}{\partial u_1}\right]\delta u_1(\tau), \tag{41}$$

$$\delta x_2(t) = F_2(t, t_1 - 1)\delta x_2(t_1). \tag{42}$$

Introducing the notation

$$Q_1(t,\tau) = F_1(t,\tau)\frac{\partial f_1(\tau, x_1(\tau), u_1(\tau))}{\partial u_1} + \sum_{s=\tau}^{t-1} F_1(t,s)\frac{\partial K_1(s,\tau,x_1(\tau),u_1(\tau))}{\partial u_1},$$

(41) is represented in the form

$$\delta x_1(t)=\sum_{\tau=t_0}^{t-1} Q_1(t,\tau)\delta u_1(\tau). \tag{43}$$

Since

$$\delta x_2(t_1)=\frac{\partial G\big(x_1(t_1)\big)}{\partial x_1}\delta x_1(t_1),$$

then, taking into account representation (43) from formula(42), we obtain that

$$\delta x_2(t\,)=F_2(t,t_1-1)\frac{\partial G\big(x_1(t_1)\big)}{\partial x_1}\sum_{\tau=t_0}^{t_1-1} Q_1(t_1,\tau)\delta u_1(\tau).$$

Hence, introducing the notation

$$Q_2(t,\tau)=F_2(t,t_1-1)\frac{\partial G\big(x_1(t_1)\big)}{\partial x_1}Q_1(t_1,\tau),$$

we have that

$$\delta x_2(t\,)=\sum_{\tau=t_0}^{t_1-1} Q_2(t\,,\tau)\delta u_1(\tau). \tag{44}$$

Now, suppose

$$Q_3(t,\tau)=F_2(t,\tau)\frac{\partial f_2\big(\tau,x_2(\tau),u_2(\tau)\big)}{\partial u_2}+\sum_{s=\tau}^{t-1} F_2(t,s)\frac{\partial K_2\big(s,\tau,x_2(\tau),u_2(\tau)\big)}{\partial u_2}.$$

By analogy with the proof of formula (43) it is proved that

$$y(t\,)=\sum_{\tau=t_1}^{t\,-1} Q_3(t\,,\tau)\delta u_2(\tau). \tag{45}$$

Using proved representations (43), (44), (45), let us transform the individual summands in inequalities (36), (38).

Using representations (43) and (44), it is proved that

$$\delta x_1'(t_1)\frac{\partial^2\varphi_1\big(x_1(t_1)\big)}{\partial x_1^2}\delta x_1(t_1)=\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u_1'(\tau)Q_1'(t_1,\tau)\frac{\partial^2\varphi_1\big(x_1(t_1)\big)}{\partial x_1^2}Q_1(t_1,s)\,\delta u_1(s), \tag{46}$$

$$\delta x_1'(t_1)\frac{\partial^2 M\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1^2}\delta x_1(t_1)=$$

$$=\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u_1'(\tau)Q_1'(t_1,\tau)\frac{\partial^2 M\big(\psi_2(t_1-1),x_1(t_1)\big)}{\partial x_1^2}Q_1(t_1,s)\,\delta u_1(s), \tag{47}$$

$$\sum_{t=t_0}^{t_1-1}\delta x_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial x_1^2}\delta x_1(t)=$$

$$=\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u_1'(\tau)\left[\sum_{\max(\tau,s)+1}^{t_1-1} Q_1'(t,\tau)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial x_1^2}Q_1(t,s)\right]\delta u_1(s), \tag{48}$$

$$\sum_{t=t_0}^{t_1-1}\delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1\partial x_1}\delta x_1(t)=$$

$$= \sum_{t=t_0}^{t_1-1} \delta u_1'(t) \left[\sum_{\tau=t_0}^{t-1} \frac{\partial^2 H_1(t, x_1(t), u_1(t), \psi_1(t))}{\partial u_1 \partial x_1} Q_1(t,\tau) \right] \delta u_1(\tau), \quad (49)$$

$$\sum_{t=t_1}^{t_2-1} \delta x_2'(t) \frac{\partial^2 H_2(t, x_2(t), u_2(t), \psi_2(t))}{\partial x_2^2} \delta x_2(t) =$$

$$= \sum_{\tau=t_0}^{t-1} \sum_{s=t_0}^{t-1} \delta u_2'(\tau) \left[\sum_{t=t_1}^{t_2-1} Q_2'(t,\tau) \frac{\partial^2 H_1(t, x_2(t), u_2(t), \psi_2(t))}{\partial x_2^2} Q_2(t,s) \right] \delta u_2(s). \quad (50)$$

Further, using representation (45) it is proved that

$$y'(t_2) \frac{\partial^2 \varphi_2(x_2(t_2))}{\partial x_2^2} y(t_2) = \sum_{\tau=t_1}^{t_2-1} \sum_{s=t_0}^{t_2-1} \delta u_2'(\tau) Q_3'(t_2,\tau) \frac{\partial^2 \varphi_2(x_2(t_2))}{\partial x_2^2} Q_3(t_2,s)\, \delta u_2(s), \quad (51)$$

$$\sum_{t=t_1}^{t_2-1} \delta u_2'(t) \frac{\partial^2 H_1(t, x_2(t), u_2(t), \psi_2(t))}{\partial u_2 \partial x_2} y(t) =$$

$$= \sum_{t=t_1}^{t_2-1} \delta u_2'(t) \left[\sum_{\tau=t_1}^{t-1} \frac{\partial^2 H_1(t, x_2(t), u_2(t), \psi_2(t))}{\partial u_2 \partial x_2} Q_3(t,\tau)\, \delta u_2(\tau) \right], \quad (52)$$

$$\sum_{t=t_1}^{t_2-1} y'(t) \frac{\partial^2 H_2(t, x_2(t), u_2(t), \psi_2(t))}{\partial x_2^2} y(t) =$$

$$= \sum_{\tau=t_1}^{t_2-1} \sum_{s=t_1}^{t_2-1} \delta u_2'(\tau) \left[\sum_{\max(\tau,s)+1}^{t_2-1} Q_3'(t,\tau) \frac{\partial^2 H_1(t, x_2(t), u_2(t), \psi_2(t))}{\partial x_2^2} Q_3(t,s) \right] \delta u_2(s). \quad (53)$$

Introduce the notations

$$D_1(\tau,s) = -Q_1'(t_1,\tau) \frac{\partial^2 \varphi_1(x_1(t_1))}{\partial x_1^2} Q_1(t_1,s) - Q_2'(t_1,\tau) \frac{\partial^2 \varphi_2(x_2(t_2))}{\partial x_2^2} Q_2(t_1,s) +$$
$$+ Q_1'(t_1,\tau) \frac{\partial^2 M(\psi_2(t_1-1), x_1(t_1))}{\partial x_1^2} Q_1(t_1,s) +$$
$$+ \sum_{\max(\tau,s)+1}^{t_1-1} Q_1'(t,\tau) \frac{\partial^2 H_1(t, x_1(t), u_1(t), \psi_1(t))}{\partial x_1^2} Q_1(t,s), \quad (54)$$

$$D_2(\tau,s) = -Q_3'(t_2,\tau) \frac{\partial^2 \varphi_2(x_2(t_2))}{\partial x_2^2} Q_3(t_2,s) +$$
$$+ \sum_{\max(\tau,s)+1}^{t_2-1} Q_3'(t,\tau) \frac{\partial^2 H_1(t, x_2(t), u_2(t), \psi_2(t))}{\partial x_2^2} Q_3(t,s). \quad (55)$$

Taking into account identities (46)-(53) and notations (54), (55), inequalities (36), (38) are represented in the following form:

$$\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u_1'(\tau)D_1(\tau,s)\,\delta u_1(s) + 2\sum_{t=t_0}^{t_1-1}\left[\sum_{\tau=t_0}^{t-1}\delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1\partial x_1}Q_1(t,\tau)\,\delta u_1(\tau)\right]+ + \sum_{t=t_0}^{t_1-1}\delta u_1'(t)\frac{\partial^2 H_1\big(t,x_1(t),u_1(t),\psi_1(t)\big)}{\partial u_1^2}\delta u_1(t)\le 0, \quad (56)$$

$$\sum_{\tau=t_1}^{t_2-1}\sum_{s=t_1}^{t_2-1}\delta u_2'(\tau)D_2(\tau,s)\,\delta u_2(s) + 2\sum_{t=t_1}^{t_2-1}\left[\sum_{\tau=t_1}^{t-1}\delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2\partial x_2}Q_3(t,\tau)\,\delta u_2(\tau)\right]+ + \sum_{t=t_1}^{t_2-1}\delta u_2'(t)\frac{\partial^2 H_2\big(t,x_2(t),u_2(t),\psi_2(t)\big)}{\partial u_2^2}\delta u_2(t)\le 0. \quad (57)$$

Let us formulate the obtained result.

Theorem 2. The optimality of the classical extremal $\big(u_1(t),u_2(t)\big)$ requires that inequalities (56) and (57) hold for all $\delta u_1(t)\in R^r, t\in T_1$ and $\delta u_2(t)\in R^q, t\in T_2$, respectively.

Inequalities (56) and (57) are fairly general and constructively verifiable necessary second-order optimality conditions.

From these we can obtain relatively easily verifiable necessary second-order optimality conditions. But they will be less informative than the result of Theorem 2.

Here is one of them.

Theorem 3. The optimality of the classical extremal $\big(u_1(t),u_2(t)\big)$ requires that the inequalities

$$v_1{}'D_1(\theta,\theta)v_1+v_1{}'\frac{\partial^2 H_1\big(\theta,x_1(\theta),u_1(\theta),\psi_1(\theta)\big)}{\partial u_1^2}v_1\le 0, \quad (58)$$

$$v_2{}'D_2(\theta,\theta)v_2+v_2{}'\frac{\partial^2 H_2\big(\theta,x_2(\theta),u_2(\theta),\psi(\theta)\big)}{\partial u_2^2}v_2\le 0 \quad (59)$$

hold for all $v_1\in R^r$ and $\theta\in T_1$ and $v_2\in R^q,\theta\in T_2$, respectively.

Note that inequalities (58) and (59) are analogs of the Gabasov-Kirillova optimality condition obtained by them in [6, 8] for the case of the problem of optimal control of ordinary differential equations in a different way.

6. Conclusion

In this study, we investigate a discrete two-stage control problem described by two nonlinear Volterra-type difference equations. Using one variant of the method of increments, implicit general necessary first- and second-order optimality conditions are first obtained. An analog of the Euler equation is proved.

Using a representation of the solutions of the linearized equations, we were able to obtain the general necessary second-order optimality condition in explicit form.

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