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# On a discrete nonlocal optimal control problem with a non-typical quality criterion

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**ABSTRACT**

*We investigate a discrete optimal control problem described by an ordinary difference equation with nonlocal boundary conditions and a Moiseyev-type functional (with an untyped quality criterion). Under various assumptions, the necessary first-order optimality conditions are proved.*

## 1. Introduction

Various optimal control problems described by ordinary difference equations with a local initial condition have been studied in [1-5] and others.

In [6-8], optimal control problems described by ordinary difference equations with nonlocal boundary conditions are studied and a number of necessary optimality conditions are proved under various assumptions.

In this study, we also address a nonlocal optimal control problem described by ordinary difference equations, but the quality functional is more general (untyped) than the quality criteria found in [6-8].

The quality functional considered in this study is a discrete analog of the quality functional introduced in [9] by N.N. Moiseyev.

A number of necessary first-order optimality conditions are obtained in this work.

As noted, for example, in [10], two approaches to the study of optimal control problems of discrete dynamic systems are purely technically possible. In the first approach, the problem of optimal control of a discrete system is reduced to a mathematical programming problem with equality-type constraints of a special kind (see, e.g., [11-13]), which is investigated using the techniques and methods of mathematical programming, and then the obtained results are interpreted in terms of the original control system. In the second approach (see, e.g., [1-5, 10, 14, 15]), the problem of optimal control of a discrete system is initially considered as a variational problem, i.e., as an optimization problem on the set of trajectories of a dynamic system. And its study is completely based on the ideas and methods of variational calculus and optimal control theory.

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Based on the above, this study follows the second approach outlined above.

## 2. Problem statement

Suppose that $T = \{t_0, t_0 + 1, \ldots, t_1 - 1\}$ is a specified finite sequence of natural numbers, $U \subset R^r$ is a specified non-empty, open and bounded set.

We shall consider the problem of finding the minimum value of the functional

$$J(u) = \varphi\big(x(t_0), x(t_1)\big) + \sum_{t=t_0}^{t_1-1} \sum_{s=t_0}^{t_1-1} F\big(t, s, x(t), x(s)\big) \tag{1}$$

under the constraints

$$u(t) \in U \subset R^r, t \in T, \tag{2}$$

$$x(t+1) = f\big(t, x(t), u(t)\big), t \in T, \tag{3}$$

$$L_0 x(t_0) + L_1 x(t_1) = \ell. \tag{4}$$

Here, $f(t, x, u)$ is a specified $n$-dimensional vector function that is continuous in the set of variables together with partial derivatives in $(x, u)$, $L_0$, $L_1$ are specified $(n \times n)$-dimensional constant matrices,

$\varphi(x_0, x_1)$ is a specified continuously differentiable scalar function, $F(t, s, a, b)$ is a specified scalar function, continuous in the set of variables together with partial derivatives in $(a, b)$, $\ell$ is a specified $n$-dimensional constant vector, $u(t)$ is an $r$- dimensional, discrete and bounded vector function of control actions (admissible control).

It is assumed that for each given admissible control u(t), boundary value problem (3)-(4) has a unique discrete solution.

The admissible control $u(t)$ that affords the minimum value of functional (1) under constraints (2)-(4) will be called an optimal control, and the corresponding process $\big(u(t), x(t)\big)$ an optimal process.

The aim of the study is to obtain necessary first-order optimality conditions in the problem under investigation.

## 3. Calculating variations of the quality criterion and an analogue of Euler's equation

Suppose that $u(t)$ and $\bar{u}(t) = u(t) + \Delta u(t)$ are admissible controls.

Denote their corresponding solutions of boundary value problem (3)-(4) by $x(t)$ and $\bar{x}(t) = x(t) + \Delta x(t)$.

The increment of functional (1) corresponding to the increment $\Delta u(t)$ of the control $u(t)$ will be written as follows:

$$\Delta J(u) = J(\bar{u}) - J(u) = \varphi\big(\bar{x}(t_0), \bar{x}(t_1)\big) - \varphi\big(x(t_0), x(t_1)\big) + \\ + \sum_{t=t_0}^{t_1-1} \sum_{s=t_0}^{t_1-1} F\big(t, s, \bar{x}(t), \bar{x}(s)\big) - \sum_{t=t_0}^{t_1-1} \sum_{s=t_0}^{t_1-1} F\big(t, s, x(t), x(s)\big). \tag{5}$$

It is clear from the introduced notation that the increment $\Delta x(t)$ of the trajectory $x(t)$ is the solution to the boundary value problem

$$\Delta x(t+1) = f\big(t, \bar{x}(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big), \tag{6}$$

$$L_0 \Delta x(t_0) + L_1 \Delta x(t_1) = 0. \tag{7}$$

Suppose that $\psi(t)$ is an as yet arbitrary $n$-dimensional vector function, and $\lambda \in R^n$ is an

unknown constant vector.

From formula (6) we get that

$$\psi'(t_1-1)\Delta x(t_1)-\psi'(t_0-1)\Delta x(t_0)+\sum_{t=t_0}^{t_1-1}\psi'(t-1)\Delta x(t)=$$

$$=\sum_{t=t_0}^{t}\psi'(t)\Big(f\big(t,\bar{x}(t),\bar{u}(t)\big)-f\big(t,x(t),u(t)\big)\Big), \tag{8}$$

We will introduce an analogue of the Hamilton-Pontryagin function

$$H\big(t,x(t),u(t),\psi(t)\big)=\sum_{t=t}^{t_1-1}\psi'(t)f\big(t,x(t),u(t)\big).$$

Taking into account the expression of the Hamilton-Pontryagin function and identity (8), increment (5) of functional (1) is written in the form

$$\Delta J(u)=\varphi\big(\bar{x}(t_0),\bar{x}(t_1)\big)-\varphi\big(x(t_0),x(t_1)\big)+$$

$$+\sum_{t=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\Big(F\big(t,s,\bar{x}(t),\bar{x}(s)\big)-F\big(t,s,x(t),x(s)\big)\Big)+$$

$$+\lambda' L_0\Delta x(t_0)+\lambda' L_1\Delta x(t_1)+\psi'(t_1-1)\Delta x(t_1)-\psi'(t_0-1)\Delta x(t_0)+\sum_{t=t_0}^{t_1-1}\psi'(t-1)\Delta x(t)-$$

$$-\sum_{t=t_0}^{t_1-1}\big[H\big(t,\bar{x}(t),\bar{u}(t),\psi(t)\big)-H\big(t,x(t),u(t),\psi(t)\big)\big]. \tag{9}$$

Taking into account the smoothness conditions imposed on the functions $\varphi(x_0,x_1), f(t,x,u), F(t,s,a,b)$, using Taylor's formula we obtain that

$$F\big(t,s,\bar{a},\bar{b}\big)-F(t,s,a,b)=\frac{\partial F'(t,s,a,b)}{\partial a}\Delta a+\frac{\partial F'(t,s,a,b)}{\partial b}\Delta b+o_1([\|\Delta a\|+\|\Delta b\|]),$$

$$H(t,\bar{x},\bar{u},\psi)-H(t,x,u,\psi)=\frac{\partial H'(t,x,u,\psi)}{\partial x}\Delta x+\frac{\partial H'(t,x,u,\psi)}{\partial u}\Delta u+o_2([\|\Delta x\|+\|\Delta u\|]),$$

$$\varphi(\bar{x}_0,\bar{x}_1)-\varphi(x_0,x_1)=\frac{\partial\varphi'(x_0,x_1)}{\partial x_0}\Delta x_0+\frac{\partial\varphi'(x_0,x_1)}{\partial x_1}\Delta x_1+o_3([\|\Delta x_0\|+\|\Delta x_1\|]). \tag{10}$$

Here, $\|\alpha\|$ is the norm of the vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)'$, determined by the formula $\|\alpha\|=\sum_{i=1}^{n}|\alpha_i|$, and $o(\alpha)$ is an infinitesimal quantity of higher order than $\alpha$, i.e. $o(\alpha)/\alpha\to 0$ at $\alpha\to 0$.

Taking into account formula (10) in the formula of increment (9) we obtain that

$$\Delta J(u)=\frac{\partial\varphi'\big(x(t_0),x(t_1)\big)}{\partial x(t_0)}\Delta x(t_0)-\frac{\partial\varphi'\big(x(t_0),x(t_1)\big)}{\partial x(t_1)}\Delta x(t_1)+$$

$$+\sum_{t=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\Big[\frac{\partial F'\big(t,s,x(t),x(s)\big)}{\partial a}\Delta x(t)+\frac{\partial F'\big(t,s,x(t),x(s)\big)}{\partial b}\Delta x(s)+$$

$$+o_1([\|\Delta x(t)\|+\|\Delta x(s)\|])]-$$

$$-\sum_{t=t_0}^{t_1-1}\Big[\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial x}\Delta x(t)-\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial u}\Delta u(t)+$$

$$+o_2([\|\Delta x(t)\|+\|\Delta u(t)\|])]+\psi'(t_1-1)\Delta x(t_1)-\psi'(t_0-1)\Delta x(t_0)+$$

$$+\lambda' L_0 \Delta x(t_0) + \lambda' L_1 \Delta x(t_1) + \sum_{t=t_0}^{t_1-1} \psi'(t-1)\Delta x(t). \quad (11)$$

Now suppose that $\psi(t)$ and $\psi(t)$satisfy the relations (i.e., $\psi(t)$ is a solution of the boundary value problem)

$$\psi(t-1) = \frac{\partial H\big(t, x(t), u(t), \psi(t)\big)}{\partial x} -$$
$$- \sum_{s=t_0}^{t_1-1} \left[ \frac{\partial F'\big(t, s, x(t), x(s)\big)}{\partial a} + \frac{\partial F'\big(s, t, x(s), x(t)\big)}{\partial b} \right], \quad (12)$$

$$\psi(t_1 - 1) = -\frac{\partial \varphi\big(x(t_0), x(t_1)\big)}{\partial x_1} - L_1{}'\lambda,$$

$$\psi(t_0 - 1) = \frac{\partial \varphi\big(x(t_0), x(t_1)\big)}{\partial x_0} + L_0{}'\lambda. \quad (13)$$

Boundary value problem (12)-(13) is an analogue of the adjoint system (see, e.g., [1]) for the optimal control problem under consideration.

Then formula of increment (11), when simplified, will take the form

$$\Delta J(u) = -\sum_{t=t_0}^{t_1-1} \frac{\partial H'\big(t, x(t), u(t), \psi(t)\big)}{\partial u} \Delta u(t) - \sum_{t=t_0}^{t_1-1} \sum_{s=t_0}^{t_1-1} o_1([\|\Delta x(t)\| + \|\Delta x(s)\|]) -$$

$$- \sum_{t=t_0}^{t_1-1} o_2([\|\Delta x(t)\| + \|\Delta u(t)\|]) + o_3([\|\Delta x(t_0)\| + \|\Delta x(t_1)\|]). \quad (14)$$

Considering boundary value problem (6)-(7), we obtain that the increment $\Delta x(t)$ of the trajectory $x(t)$ is a solution of the linearized problem

$$\Delta x(t+1) = \frac{\partial f\big(t, x(t), u(t)\big)}{\partial x} \Delta x(t) + \frac{\partial f\big(t, x(t), u(t)\big)}{\partial u} \Delta u(t) +$$
$$+ o_4\big(\|\Delta x(t)\| + \|\Delta u\|(t)\big), \quad (15)$$

$$L_0 \Delta x(t_0) + L_1 \Delta x(t_1) = 0. \quad (16)$$

Suppose that $\varepsilon$ is an arbitrary sufficiently small in absolute value number, and $\delta u(t) \in R^r, t \in T$ is an arbitrary discrete, bounded vector function.

By assumption, the control domain $U$ is open. Therefore, the special increment of the admissible control $u(t)$ can be determined from the formula

$$\Delta u(t; \varepsilon) = \varepsilon \delta u(t), t \in R^r. \quad (17)$$

Denote the special increment of the admissible trajectory $x(t)$ by $\Delta x(t; \varepsilon)$.

Suppose that

$$\Delta x(t; \varepsilon) = \varepsilon \delta x(t) + o(t; \varepsilon). \quad (18)$$

Here, $\delta x(t)$ (admissible trajectory variation) is an as yet unknown *n*-dimensional function to be defined.

Using formulas (17) and (18), by means of linearized boundary value problem (15)-(16), we prove

**Lemma 1.** The variation $\delta x(t)$ of the trajectory $x(t)$ is the solution to the boundary value problem

$$\delta x(t+1) = f_x\big(t, x(t), u(t)\big), t \in T,$$

$$L_0\delta x(t_0) + L_1\delta x(t_1) = 0.$$

Considering formulas (17) and (18), we obtain from the increment formula (14) that

$$J(u(t)+\varepsilon\delta u) - J\big(u(t)\big) = -\varepsilon\sum_{t=t_0}^{t_1-1}\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial u}\delta u(t) + o(\varepsilon).$$

Consequently, the first variation of functional (1) has the form

$$\delta^1 J(u:\delta u) = \sum_{t=t_0}^{t_1-1}\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial u}\delta u(t). \tag{19}$$

By virtue of the openness of the control domain, the first variation of the functional must be zero.

Therefore, given formula (19) and the arbitrariness of the admissible variation $\delta u(t)$ of the control, we arrive at the following statement.

**Theorem 1.** The optimality of the admissible control $u(t)$ in this problem requires that the condition

$$\frac{\partial H'\big(\theta,x(\theta),u(\theta),\psi(\theta)\big)}{\partial u} = 0 \tag{20}$$

hold for all $\theta \in T$.

This optimality condition (20) is an analogue of Euler's equation for the problem under consideration (see, e.g., [1, 4, 5]).

## 4. Linearized optimality condition

Suppose that the set $U$ is convex. Then the special increment of the admissible control $u(t)$ can be determined from the formula

$$\Delta u(t;\mu) = \mu\big(v(t) - u(t)\big). \tag{21}$$

Here $\mu \in [0,1]$ is an arbitrary number, and $v(t) \in U -$ is an arbitrary admissible control.

Denote by $\Delta x(t;\mu)$ special increment of the trajectory $x(t)$, corresponding to special increment (21) of the control $u(t)$.

By analogy with the proof of Lemma 1 we prove

**Lemma 2.** If the control domain $U$ is convex in the considered problem for the special increment $\Delta x(t;\mu)$ of the trajectory $x(t)$ the following representation takes place:

$$\Delta x(t;\mu) = \mu q(t) + o(t;\mu).$$

Here, $q(t)$ is a solution of the problem

$$q(t+1) = \frac{\partial f\big(t,x(t),u(t)\big)}{\partial x}\Delta q(t) + \frac{\partial f\big(t,x(t),u(t)\big)}{\partial u}\big(v(t)-u(t)\big), \tag{22}$$

$$L_0 q(t_0) + L_1 q(t_1) = 0.$$

Considering Lemma 2 and formulas (21), (22), increment (14) of functional (1) is written in the form

$$J\Big(u(t)+\mu\big(v(t)-u(t)\big)\Big) - J\big(u(t)\big) =$$

$$= -\mu\sum_{t=t_0}^{t_1-1}\frac{\partial H'\big(t,x(t),u(t),\psi(t)\big)}{\partial u}\big(v(t)-u(t)\big) + o(\mu). \tag{23}$$

Suppose $u(t)$ is an optimal control.

Then it follows from expansion (23) that

$$-\mu\sum_{t=t_0}^{t_1-1}\frac{\partial H'\left(t,x(t),u(t),\psi(t)\right)}{\partial u}\left(v(t)-u(t)\right)+o(\mu)\geq 0.$$

From this inequality follows

**Theorem 2.** If the set $U$ is convex, the optimality of the admissible control $u(t)$ requires that the inequality

$$\sum_{t=t_0}^{t_1-1}\frac{\partial H'\left(t,x(t),u(t),\psi(t)\right)}{\partial u}\left(v(t)-u(t)\right)\leq 0 \qquad (24)$$

hold for all admissible controls $v(t)$.

Inequality (24), being a necessary first-order optimality condition, is an analogue of the linearized (differential) maximum condition (see, e.g., [1-5]).

Using arbitrariness of the admissible controls $v(t)$ from Theorem 2 we obtain

**Theorem 3.** If the set $U$ is convex, the optimality of the admissible control $u(t)$ requires that the inequality

$$\frac{\partial H'\left(\theta,x(\theta),u(\theta),\psi(\theta)\right)}{\partial u}\left(w-u(\theta)\right)\leq 0$$

hold for all $w\in U$ and $\theta\in T$.

Thus, we obtain a pointwise necessary optimality condition.

## 5. Conclusion

In this study we consider a nonlocal discrete optimal control problem with an untyped quality criterion. Interpreting the problem under consideration as a discrete analogue of a non-classical problem of variational calculus and applying different variations of admissible control under different smoothness conditions on the parameters of the optimal control problem under investigation, a number of necessary optimality conditions in the form of a linearized maximum condition and an analogue of Euler's equation are established.

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