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# On the fourth order moment of the renewal-reward process with dependent components

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ABSTRACT

*This paper investigates the 4th moment of renewal-reward process with dependent components, expanding upon classical renewal theory by incorporating dependency structures among the components. We goal to derive expressions for the 4th moment and explore how dependencies influence the behavior of the renewal-reward process.*

## 1. Introduction

Let random vectors $(\xi_n, \eta_n), n \geq 1$ be independent and identically distributed. In the general case, the random variable $\eta_n$ is assumed to depend on the random variable $\xi_n$. Let's denote the distribution function of $\xi_n$ by $F$: $F(x) = P\{\xi_n \leq x\}$.

Let's introduce the following sum:

$$S_{\nu(t)} = \sum_{n=1}^{\nu(t)} \eta_n, \tag{1.1}$$

where $\nu(t) = \max\{n: T_n \leq t\}$, $t > 0$ is the renewal process and $T_n = \sum_{i=1}^{n} \xi_i$, $n = 1,2,....$ The process $S_{\nu(t)}$, $t \geq 0$ is called as renewal-reward process and represents the sum of the rewards obtained up to time $t$. Note that, renewal-reward processes are applied in stochastic control theory, insurance and reliability theories, analyzing systems and processes in areas like operations research, economics, finance, and engineering.(see, [1]-[9]). In the present study, the asymptotic expansion for the fourth-order initial moment of the renewal-reward process is investigated.

## 2. Problem statement

Denote the 4th order moment of the renewal-reward process (1.1) by $D_4(t)$ and the mathematical expectation of the renewal-reward process by $M_n(t)$ when the rewards are given by $\eta_k^n$ $(k \geq 1, n \geq 1)$:

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$$D_4(t) = E\left(S_{\nu(t)}\right)^4 = E\left(\sum_{k=1}^{\nu(t)} \eta_k\right)^4, \qquad M_n(t) = E\left(\sum_{k=1}^{\nu(t)} \eta_k^n\right).$$

It is clear that, $D_1(t) = M_1(t) = D(t)$, where $D(t)$ is the mathematical expectation of the process (1.1).

Our main purpose is to obtain an asymptotic expansion for $D_4(t)$ as $t \to \infty$. For this, we will obtain exact formula for $D_4(t)$ in terms of $M_1(t)$, $M_2(t)$, $M_3(t)$ and $M_4(t)$. Then using asymptotic expansions for $M_n(t), n = \overline{1,4}$ will take us to our main purpose.

### 2.1. Exact formula for $D_4(t)$ in terms of $M_n(t), n = \overline{1,4}$

Let $F^{*(k)}$ be the $k$-th convolution of $F$, and let $f_k(t) = \frac{dF^{*(k)}(t)}{dH(t)}$, where $H(t) = E\big(\nu(t)\big)$ is a renewal function. Since

$$H(t) = \sum_{k=1}^{\infty} F^{*(k)}(t),$$

it follows that $H(t) = 0$ implies that all $F^{*(k)}(t) = 0$; thus $F^{*(k)}(t) \ll H(t)$, and $f_k(t)$ is well defined.

Denote

$$M_1 * M_3(t) = \int_0^t M_1(t-x)dM_3(x)\,, M_2 * M_1^{*(2)}(t) = \int_0^t M_1^{*(2)}(t-x)d\,M_2(x),$$

$$M_2^{*(2)}(t) = \int_0^t M_2^{*(1)}(t-x)dM_2(x)\,, M_1^{*(4)}(t) = \int_0^t M_1^{*(3)}(t-x)dM_1(x).$$

To obtain exact formula for $D_4(t)$ in terms of $M_1(t)$, $M_2(t)$, $M_3(t)$ and $M_4(t)$, we first need to prove the following lemma.

**Lemma 2.1.** Let random vectors $(\xi_n, \eta_n), n \geq 1$ be independent and identically distributed. In the general case, the random variable $\eta_n$ is assumed to depend on the random variable $\xi_n$. Then,

$$E\left(\sum_{\substack{j \neq k \\ j,k \leq \nu(t)}} \eta_j^3 \eta_k\right) = 2M_3 * M_1(t);\ E\left(\sum_{j<k\leq\nu(t)} \eta_j^2\eta_k^2\right) = M_2^{*(2)}(t),$$

$$E\left(\sum_{\substack{i\neq j \\ i\neq k \\ j<k\leq\nu(t) \\ i\leq\nu(t)}} \eta_i^2\eta_j\eta_k\right) = 3M_2 * M_1^{*(2)}(t);\ E\left(\sum_{i<j<k<l\leq\nu(t)} \eta_i\eta_j\eta_k\eta_l\right) = M_1^{*(4)}(t).$$

**Proof.** It's not difficult to see, that

$$M_1(t) = E\left(\sum_{k=1}^{\nu(t)} \eta_k\right) = E\left(\sum_{k=1}^{\infty} \eta_k I\{T_k \leq t\}\right) =$$

$$= \sum_{k=1}^{\infty} \int_0^t E(\eta_k | T_k = s)dF^{*(k)}(s) = \int_0^t \left(\sum_{k=1}^{\infty} E(\eta_k|T_k = s)f_k(s)\right)dH(s).$$

By the similar way

$$M_2(t) = \int_0^t \left( \sum_{k=1}^{\infty} E(\eta_k^2 | T_k = s) f_k(s) \right) dH(s),$$

$$M_3(t) = \int_0^t \left( \sum_{k=1}^{\infty} E(\eta_k^3 \big| T_k = s) f_k(s) \right) dH(s).$$

Therefore

$$\frac{dM_n(s)}{dH(s)} = \sum_{k=1}^{\infty} E(\eta_k^n | T_k = s) f_k(s)\,, n = \overline{1{,}3}.$$

It can be seen that,

$$E\left( \sum_{k<j\le\nu(t)} \eta_j^3 \eta_k \right) = M_1 * M_3(t), \qquad E\left( \sum_{j<k\le\nu(t)} \eta_j^2 \eta_k^2 \right) = M_2^{*(2)}(t).$$

So,

$$E\left( \sum_{\substack{j\neq k \\ j,k\le\nu(t)}} \eta_j^3 \eta_k \right) = E\left( \sum_{j<k\le\nu(t)} \eta_j^3 \eta_k \right) + E\left( \sum_{k<j\le\nu(t)} \eta_j^3 \eta_k \right) = 2M_3 * M_1(t).$$

Next

$$E\left( \sum_{i<j<k\le\nu(t)} \eta_i^2 \eta_j \eta_k \right) = M_2 * M_1^{*(2)}(t).$$

By the similar way,

$$E\left( \sum_{j<i<k\le\nu(t)} \eta_i^2 \eta_j \eta_k \right) = \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} M_1(t-w_2) dM_2(w_2-w_1) dM_1(w_1) = M_1 * M_2 * M_1$$
$$= M_2 * M_1^{*(2)}(t),$$

$$E\left( \sum_{j<k<i\le\nu(t)} \eta_i^2 \eta_j \eta_k \right) = \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} M_1(t-w_2) dM_1(w_2-w_1) dM_2(w_1) = M_1 * M_1 * M_2$$
$$= M_2 * M_1^{*(2)}(t).$$

So,

$$E\left( \sum_{\substack{i\neq j \\ i\neq k \\ j<k\le\nu(t) \\ i\le\nu(t)}} \eta_i^2 \eta_j \eta_k \right) = 3M_2 * M_1^{*(2)}(t).$$

by the similar way

$$E\left( \sum_{i<j<k<l\le\nu(t)} \eta_i \eta_j \eta_k \eta_l \right) = M_1^{*(4)}(t).$$

this completes the proof of Lemma 2.1.

We can easily get that lemma 2.2 from using proof of this lemma.

**Lemma 2.2.** Let random vectors $(\xi_n, \eta_n), n \geq 1$ be independent and identically distributed. In the general case, the random variable $\eta_n$ is assumed to depend on the random variable $\xi_n$. Then,

$$D_4(t) = 24M_1^{*(4)}(t) + 36M_2 * M_1^{*(2)}(t) + 6M_2^{*(2)}(t) + 8M_3 * M_1(t) + M_4(t).$$

### 2.2. Asymptotic expansion for $D_4(t)$

Define $R(t) = H(t) - \frac{1}{\mu_1}t - \frac{\mu_2}{2\mu_1^2} + 1$, where $\mu_k = E\xi_1^k,\ k \geq 0$

Let $r_k = \int_0^\infty t^k R(t)dt,\ k \geq 0$ and $\int_0^\infty t^k|R(t)|dt < \infty$.

**Definition 2.1.** A distribution function $F$ is said to belong to the class $\vartheta$ if some convolution of $F$ has an absolutely continuous component.

To obtain an asymptotic expansion for $D_4(t)$ we need to prove the following lemmas.

**Lemma 2.3.** If $F \in \vartheta$ and $\mu_5 < \infty$, then

$$r_1 = -\frac{\mu_2^3}{8\mu_1^4} + \frac{\mu_2\mu_3}{6\mu_1^3} - \frac{\mu_4}{24\mu_1^2};\ \ r_2 = \frac{\mu_2^4}{8\mu_1^5} - \frac{\mu_3\mu_2^2}{4\mu_1^4} + \frac{\mu_3^2}{18\mu_1^3} + \frac{\mu_2\mu_4}{12\mu_1^3} - \frac{\mu_5}{60\mu_1^2}.$$

**Proof.** We will use similar method by [3, pp.304-306]. From the conditions it is implied that [9, p.2]

$$\text{(i)} \qquad \lim_{t\to\infty} t^3R(t) = 0;$$

$$\text{(ii)} \qquad \int_0^\infty t^k|R(t)|dt < \infty, k = 0, 1, 2.$$

[3, pp. 305-306] obtained the expression for $r_0$:

$$r_0 = \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2},$$

For $r_2$ consider the functions

$$g_1(t) = t^2 \int_{x=0}^{t} xR(t-x)dF(x)\,, g_2(t) = t\int_{x=0}^{t} x^2R(t-x)dF(x),$$

$$g_3(t) = \int_{x=0}^{t} x^3R(t-x)dF(x).$$

By (ii) we can write

$$\int_0^\infty |g_1(t)|dt \leq \int_{t=0}^\infty t^2 \int_{x=0}^{t} x|R(t-x)|dF(x)\,dt = \int_{x=0}^\infty x \int_{t=x}^\infty t^2|R(t-x)|dt\,dF(x) =$$

$$= \mu_1\int_0^\infty t^2|R(t)|dt + 2\mu_2\int_0^\infty t|R(t)|dt + \mu_3\int_0^\infty |R(t)|dt < \infty.$$

By the similar way

$$\int_0^\infty |g_2(t)|dt < \infty.$$

thus $g_1$, $g_2$ and $g_3$ are integrable and

$$\int_0^\infty g_1(t)dt = \mu_1r_2 + 2\mu_2r_1 + \mu_3r_0, \qquad \int_0^\infty g_2(t)dt = \mu_2r_1 + \mu_3r_0,$$

$$\int_0^\infty g_3(t)dt = \mu_3 r_0.$$

Also, using the same method in [3, p.305] it can be shown that $g_1$, $g_2$ and $g_3$ are of bounded variation on $[0,\infty)$. Thus, $g_1$, $g_2$ and $g_3$are directly Riemann integrable.

Now

$$H(t) = F(t) + \int_0^t H(t-x)dF(x).$$

Subtract $\frac{1}{\mu_1}t + \frac{\mu_2}{2\mu_1^2} - 1$ from both sides and then multiply by $t^3$ to obtain

$$t^3R(t) = \int_0^t (t-x)^3R(t-x)dF(x) + Z(t),$$

where

$$Z(t) = 3g_1(t) - 3g_2(t) + g_3(t) + \frac{1}{\mu_1}t^3\int_t^\infty \big(1-F(x)\big)dx - \frac{\mu_2}{2\mu_1^2}t^3\big(1-F(t)\big).$$

By the key renewal theorem

$$\lim_{t\to\infty} t^3R(t) = \frac{1}{\mu_1}\int_0^\infty Z(t)dt = \frac{1}{\mu_1}\left(3\mu_1 r_2 + 3\mu_2 r_1 + \mu_3 r_0 + \frac{\mu_5}{20\mu_1} - \frac{\mu_2\mu_4}{8\mu_1^2}\right)$$

But, by (i), $\lim_{t\to\infty} t^3R(t) = 0$, thus

$$r_0 = \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2},$$

$$r_1 = -\frac{\mu_2}{2\mu_1}r_0 - \frac{\mu_4}{24\mu_1^2} + \frac{\mu_2\mu_3}{6\mu_1^3} = -\frac{\mu_2^3}{8\mu_1^4} + \frac{\mu_2\mu_3}{6\mu_1^3} - \frac{\mu_4}{24\mu_1^2},$$

$$r_2 = -\frac{\mu_2}{\mu_1}r_1 - \frac{\mu_3}{3\mu_1}r_0 - \frac{\mu_5}{60\mu_1^2} + \frac{\mu_2\mu_4}{24\mu_1^3} = \frac{\mu_2^4}{8\mu_1^5} - \frac{\mu_2^2\mu_3}{4\mu_1^4} + \frac{\mu_3^2}{18\mu_1^3} + \frac{\mu_2\mu_4}{12\mu_1^3} - \frac{\mu_5}{60\mu_1^2}.$$

This completes the proof.

Define

$$\lambda_s = E\eta_1^s = \int_0^\infty E(\eta_1^s|\xi_1 = t)dF(t)\,, \qquad n_{k,s} = E\big(\xi_1^k\eta_1^s\big) = \int_0^\infty x^kE(\eta_1^s|\xi_1 = x)dF(x),$$

It is clear that $n_{k,0} = \mu_k$, $n_{0,s} = \lambda_s$ and $M_1(t) = D_1(t) = D(t)$.

Define $L(t) = D(t) - at - b$, where $a = \frac{\lambda_1}{\mu_1}$, $b = \frac{\lambda_1\mu_2}{2\mu_1^2} - \frac{n_{1,1}}{\mu_1}$

Let $l_s = \int_0^\infty t^sL(t)dt\,,\ s \geq 0,$ whenever $\int_0^\infty t^s|L(t)|dt < \infty.$

**Lemma 2.4.** If $F \in \vartheta$, $\mu_5$, $\lambda_1$ and $n_{4,1}$ exist, then

$$l_1 = \lambda_1 r_1 + n_{1,1}r_0 + \frac{n_{3,1}}{6\mu_1} - \frac{\mu_2 n_{2,1}}{4\mu_1^2}.$$

$$l_2 = \lambda_1 r_2 + 2n_{1,1}r_1 + n_{2,1}r_0 + \frac{n_{4,1}}{12\mu_1} - \frac{\mu_2 n_{3,1}}{6\mu_1^2}.$$

**Proof.** Note that since $\lambda_1$ and $n_{4,1}$ exist, then $n_{1,1}$, $n_{2,1}$ and $n_{3,1}$ also exist, because

$$E(\xi_1^3|\eta_1|) \leq \big(E(\xi_1^4|\eta_1|)\big)^{\frac{3}{4}}\big(E(|\eta_1|)\big)^{\frac{1}{4}}.$$

$$E(\xi_1^2|\eta_1|) \leq \left(E(\xi_1^3|\eta_1|)\right)^{\frac{2}{3}}\left(E(|\eta_1|)\right)^{\frac{1}{3}}.$$
$$E(\xi_1|\eta_1|) \leq \left(E(\xi_1^2|\eta_1|)\right)^{\frac{1}{2}}\left(E(|\eta_1|)\right)^{\frac{1}{2}}.$$

Also,

$$S_{\nu(t)} = \sum_{i=1}^{\nu(t)} \eta_i = \sum_{i=1}^{\nu(t)+1} \eta_i - \eta_{\nu(t)+1}.$$

Since $\nu(t)+1$ is a stopping time [7, pp.1-4]

$$D(t) = \lambda_1(H(t)+1) - E\left(\eta_{\nu(t)+1}\right).$$

Subtracting $at+b$ from both sides and multiplying by $t$ we obtain

$$t^2L(t) = \lambda_1 t^2 R(t) + t^2\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right).$$

Thus if $t^2\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right)$ is integrable then

$$l_2 = \lambda_1 r_2 + \int_0^\infty t^2\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right)dt$$

But

$$\frac{n_{1,1}}{\mu_1} = \frac{1}{\mu_1}\int_0^\infty xE(\eta_1|\xi_1 = x)dF(x) \tag{1}$$

and [7, p.134]

$$E\left(\eta_{\nu(t)+1}\right) = E\left(\eta_{\nu(t)+1}\middle|S_{\nu(t)} = 0\right)\bar{F}(t) + \int_0^t E\left(\eta_{\nu(t)+1}\middle|S_{\nu(t)} = x\right)\bar{F}(t-x)dH(x) =$$
$$= \int_t^\infty E(\eta_1|\xi_1 = x)dF(x) + \int_0^\infty E(\eta_1|\xi_1 = x)\left(H(t) - H(t-x)\right)dF(x),$$

So,

$$E\left(\eta_{\nu(t)+1}\right) = \int_t^\infty E(\eta_1|\xi_1 = x)dF(x) + \int_0^\infty E(\eta_1|\xi_1 = x)\left(H(t) - H(t-x)\right)dF(x) \tag{2}$$

It follows from (1) and (2) that

$$\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)$$
$$= \int_0^\infty E(\eta_1|\xi_1 = x)\left(R(t-x) - R(t)\right)dF(x) - \int_t^\infty E(\eta_1|\xi_1 = x)dF(x) \tag{3}$$

Thus

$$\int_0^\infty t^2\left|\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right|dt \leq \int_{x=0}^\infty\left(\int_{t=0}^\infty t^2(|R(t-x)| + |R(t)|)dt\right)E(|\eta_1||\xi_1 = x)dF(x) +$$
$$+ \int_{t=0}^\infty t^2 \int_{x=t}^\infty E(|\eta_1||\xi_1 = x)dF(x)\,dt \leq$$

$$2E|\eta_1|\int_0^\infty t^2|R(t)|dt+2E(\xi_1|\eta_1|)\int_0^\infty t|R(t)|dt+E(\xi_1^2|\eta_1|)\int_0^\infty|R(t)|dt+\frac{17}{12\mu_1}E(\xi_1^4|\eta_1|)+$$
$$+\frac{7}{3}\left(\frac{\mu_2}{2\mu_1^2}+1\right)E(\xi_1^3|\eta_1|)+\frac{1}{3}E(\xi_1^3|\eta_1|)<\infty$$

Thus $t^2\left(\frac{n_{1,1}}{\mu_1}-E\left(\eta_{\nu(t)+1}\right)\right)$ is integrable, and

$$\int_0^\infty t^2\left(\frac{n_{1,1}}{\mu_1}-E\left(\eta_{\nu(t)+1}\right)\right)dt=\int_{x=0}^\infty\left(\int_{t=0}^\infty t^2\big(R(t-x)-R(t)\big)dt\right)E(\eta_1|\xi_1=x)dF(x)-$$
$$-\int_{t=0}^\infty t^2\int_{x=t}^\infty E(\eta_1|\xi_1=x)dF(x)\,dt=$$
$$=2n_{1,1}r_1+n_{2,1}r_0-\frac{\mu_2}{6\mu_1^2}n_{3,1}+\frac{n_{4,1}}{12\mu_1}.$$

This completes the proof.

**Lemma 2.5.** Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_n$, $\mu_5$, $n_{4,n}$ exist, where $n=\max(i,j)$, $i,j\geq 1$. Then

$$M_i*M_j(t)=a_{i*j}t^2+b_{i*j}t+c_{i*j}+L_{i*j}(t),$$

where $a_{i*j}=\frac{1}{2}a_ia_j$, $b_{i*j}=a_jb_i+a_ib_j$, $c_{i*j}=b_ib_j+a_jl_{i,0}+a_il_{j,0}$, $L_{i*j}(t)=o(t^{-2}),\ t\to\infty$.

**Proof.** From the Theorem 2.1 of [2, p.139], we can write

$$M_k(t)=a_kt+b_k+L_k(t),\qquad k=i,j,$$

where $a_k=\frac{\lambda_k}{\mu_1},\ b_k=\frac{\lambda_k\mu_2}{2\mu_1^2}-\frac{n_{1,k}}{\mu_1}, L_k(t)=o(t^{-3}),\ t\to\infty$.

Also, from Lemma 2.2,

$$\int_0^\infty t^s|L_k(t)|dt<\infty,\qquad s=0,1,2;\ \ k=i,j.$$

Denote

$$l_{k,s}=\int_0^\infty t^sL_k(t)dt,\qquad s=0,1,2;\ \ k=i,j.$$

Then

$$M_i*M_j(t)=\int_0^t M_i(t-x)dM_j(x)=a_{i*j}t^2+b_{i*j}t+c_{i*j}+L_{i*j}(t),$$

where $L_{i*j}(t)=-a_j\int_t^\infty L_i(x)dx-a_i\int_t^\infty L_j(x)dx+b_iL_j(t)+\int_0^t L_i(t-x)dL_j(x)$,
$a_{i*j}=\frac{1}{2}a_ia_j$, $b_{i*j}=a_jb_i+a_ib_j$, $c_{i*j}=b_ib_j+a_jl_{i,0}+a_il_{j,0}$.

It’s not difficult to see that, as $t\to\infty$

$$\left|t^2\int_t^\infty L_k(x)dx\right|\leq\int_t^\infty t^2|L_k(x)|dx\to 0.$$

Therefore, as $t\to\infty$

$$\int_t^\infty L_k(x)dx = o(t^{-2}).$$

For the third term of $L_{i*j}(t)$, we will use Laplace Transform. Denote

$$A(t) = t^2 \int_0^t L_i(t-x)dL_j(x).$$

Denote Laplace Transform of $A(t)$ by $\hat{A}(\alpha)$:

$$\hat{A}(\alpha) = \int_0^\infty e^{-\alpha t}A(t)dt = \int_{t=0}^\infty e^{-\alpha t}t^2 \int_{x=0}^t L_i(t-x)dL_j(x)\,dt =$$

$$= \int_{x=0}^\infty \int_{t=x}^\infty e^{-\alpha t}t^2 L_i(t-x)dt\,dL_j(x) = \int_{x=0}^\infty e^{-\alpha x}dL_j(x) \int_{t=0}^\infty t^2 e^{-\alpha t}L_i(t)dt +$$

$$+2\int_{x=0}^\infty xe^{-\alpha x}dL_j(x) \int_{t=0}^\infty te^{-\alpha t}L_i(t)dt + \int_{x=0}^\infty x^2 e^{-\alpha x}dL_j(x) \int_{t=0}^\infty e^{-\alpha t}L_i(t)dt.$$

Therefore,

$$\lim_{t\to\infty} A(t) = 0.$$

Consequently, as $t \to \infty$

$$\int_0^t L_i(t-x)dL_j(x) = o(t^{-2}),\ L_{i*j}(t) = o(t^{-1}).$$

**Corollary 1.** Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_2$, $\mu_4$, $n_{3,2}$ exist. Then

$$M_1 * M_3(t) = a_{1*3}t^2 + b_{1*3}t + c_{1*3} + L_{1*3}(t),$$

where $a_{1*3} = \frac{1}{2}a_1a_3$; $b_{1*3} = a_3b_1 + a_1b_3$; $c_{1*3} = b_1b_3 + a_3l_{1,0} + a_1l_{3,0}$; $L_{1*3}(t) = o(t^{-2}),\ t \to \infty$.

**Corollary 2.** Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_k$, $\mu_5$, $n_{4,k}$ exist. Then

$$M_k^{*(2)}(t) = a_k^{*(2)}t^2 + b_k^{*(2)}t + c_k^{*(2)} + L_k^{*(2)}(t),$$

where $a_k^{*(2)} = \frac{1}{2}a_k^2$; $b_k^{*(2)} = 2a_kb_k$; $c_k^{*(2)} = b_k^2 + 2a_kl_{k,0}$;

$$L_k^{*(2)}(t) = -2a_k\int_t^\infty L_k(x)dx + b_kL_k(t) + \int_0^t L_k(t-x)dL_k(x) = o(t^{-2}), \qquad t \to \infty.$$

Let $l_{1,s}^{*(2)} = \int_0^\infty t^s L_1^{*(2)}(t)dt$, $s = 0,1$; whenever $\int_0^\infty t^s \left|L_1^{*(2)}(t)\right|dt < \infty$.

**Lemma 2.6.** Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_1$, $\mu_5$, $n_{4,1}$ exist. Then

$$l_{1,0}^{*(2)} = -2a_1l_{1,1} + 2b_1l_{1,0},$$
$$l_{1,1}^{*(2)} = -a_1l_{1,2} + 2b_1l_{1,1}.$$

**Proof.** From Corollary 2

$$L_1^{*(2)}(t) = -2a_1\int_t^\infty L_1(x)dx + b_1L_1(t) + \int_0^t L_1(t-x)dL_1(x).$$

It is easy to see that

$$\int_{t=0}^{\infty}\left|\int_{x=t}^{\infty}L_1(x)dx\right|dt \le \int_{x=0}^{\infty}|L_1(x)|\int_{t=0}^{x}dt\,dx = \int_{x=0}^{\infty}x|L_1(x)|dx<\infty.$$

Therefore,

$$\int_{0}^{\infty}|L_1(t)|dt<\infty.$$

On the other hand

$$\left|\int_{t=0}^{\infty}\int_{x=0}^{t}L_1(t-x)dL_1(x)\,dt\right| = \left|\int_{x=0}^{\infty}\int_{t=x}^{\infty}L_1(t-x)dt\,L_1(x)\right| = \left|\int_{x=0}^{\infty}dL_1(x)\int_{t=0}^{\infty}L_1(t)dt\right|<\infty.$$

For $L_1^{*(2)}(t)$ is integrable and we derive that:

$$l_{1,0}^{*(2)} = \int_{0}^{\infty}L_1^{*(2)}(t)dt = -2a_1\int_{t=0}^{\infty}\int_{x=t}^{\infty}L_1(x)dx\,dt + b_1\int_{0}^{\infty}L_1(t)dt + \int_{t=0}^{\infty}\int_{x=0}^{t}L_1(t-x)dL_1(x)\,dt =$$

$$= -2a_1\int_{x=0}^{\infty}xL_1(x)dx + b_1 l_{1,0} + \int_{x=0}^{\infty}dL_1(x)\int_{t=0}^{\infty}L_1(t)dt = -2a_1 l_{1,1} + 2b_1 l_{1,0}.$$

It easy to see that

$$\int_{t=0}^{\infty}t\left|\int_{x=t}^{\infty}L_1(x)dx\right|dt \le \int_{x=0}^{\infty}|L_1(x)|\int_{t=0}^{x}tdt\,dx \le \frac{1}{2}\int_{x=0}^{\infty}x^2|L_1(x)|dx<\infty.$$

Then

$$\int_{0}^{\infty}t|L_1(t)|dt<\infty$$

On the other hand,

$$\left|\int_{t=0}^{\infty}t\int_{x=0}^{t}L_1(t-x)dL_1(x)\,dt\right| = \left|\int_{x=0}^{\infty}\int_{t=x}^{\infty}tL_1(t-x)dt\,dL_1(x)\right| =$$

$$= \left|\int_{x=0}^{\infty}dL_1(x)\int_{t=0}^{\infty}tL_1(t)dt\right| + \left|\int_{x=0}^{\infty}xdL_1(x)\int_{t=0}^{\infty}L_1(t)dt\right|<\infty.$$

$tL_1^{*(2)}(t)$ is integrable and

$$tL_1^{*(2)}(t) = -2a_1t\int_{t}^{\infty}L_1(x)dx + b_1tL_1(t) + \int_{0}^{t}tL_1(t-x)dL_1(x).$$

Consequently,

$$l_{1,1}^{*(2)} = \int_{0}^{\infty}tL_1^{*(2)}(t)dt = -a_1\int_{x=0}^{\infty}x^2L_1(x)dx + b_1 l_{1,1} + \int_{x=0}^{\infty}dL_1(x)\int_{t=0}^{\infty}tL_1(t)dt$$
$$= -a_1 l_{1,2} + 2b_1 l_{1,1}.$$

This completes the proof.

**Lemma 2.7.** If Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_1$, $\mu_5$, $n_{4,1}$ exist. Then

$$M_1^{*(3)}(t) = a_1^{*(3)}t^3 + b_1^{*(3)}t^2 + c_1^{*(3)}t + d_1^{*(3)} + L_1^{*(3)}(t),$$

where $a_1^{*(3)} = \frac{1}{3}a_1 a_1^{*(2)}$; $b_1^{*(3)} = \frac{1}{2}a_1 b_1^{*(2)} + a_1^{*(2)} b_1$; $c_1^{*(3)} = a_1 c_1^{*(2)} + b_1 b_1^{*(2)} + 2a_1^{*(2)} l_{1,0}$;

$$d_1^{*(3)} = a_1 l_1^{*(2)} + b_1 c_1^{*(2)} + b_1^{*(2)} l_{1,0} - 2a_1^{*(2)} l_{1,1};\ L_1^{*(3)}(t) = o(t^{-1}), \qquad t \to \infty.$$

**Proof.** Since

$$M_1^{*(3)}(t) = \int_0^t \left(M_1^{*(2)}(t-x)\right) dM_1(x),$$

by using Theorem 2.1 of [2, pp.143-144] and Corollary 2, we can write

$$M_1^{*(3)}(t) = a_1^{*(3)} t^3 + b_1^{*(3)} t^2 + c_1^{*(3)} t + d_1^{*(3)} + L_1^{*(3)}(t),$$

where $a_1^{*(3)} = \frac{1}{3}a_1 a_1^{*(2)}$; $b_1^{*(3)} = \frac{1}{2}a_1 b_1^{*(2)} + a_1^{*(2)} b_1$; $c_1^{*(3)} = a_1 c_1^{*(2)} + b_1 b_1^{*(2)} + 2a_1^{*(2)} l_{1,0}$;

$$d_1^{*(3)} = a_1 l_1^{*(2)} + b_1 c_1^{*(2)} + b_1^{*(2)} l_{1,0} - 2a_1^{*(2)} l_{1,1}.$$

Let's calculate the $L_1^{*(3)}(t)$:

$$L_1^{*(3)}(t) = -a_1^{*(2)} t^2 L_1(t) - b_1^{*(2)} t L_1(t) - a_1 \int_t^\infty L_1^{*(2)}(x)dx - \left(2a_1^{*(2)} t + b_1^{*(2)}\right) \int_t^\infty L_1(x)dx +$$

$$+2a_1^{*(2)} \int_t^\infty x L_1(x)dx + \int_0^t L_1^{*(2)}(t-x) dL_1(x).$$

Since $L_1(t) = o(t^{-3})$ as $t \to \infty$, then

$$tL_1(t) = o(t^{-2}), \qquad t^2 L_1(t) = o(t^{-1}), \qquad t \to \infty.$$

Since

$$\int_0^\infty x\left|L_1^{*(2)}(x)\right| dx < \infty, \qquad \int_0^\infty x^2 |L_1(x)| dx < \infty,$$

Then as $t \to \infty$

$$\int_t^\infty L_1^{*(2)}(x)dx = o(t^{-1}), \qquad \int_t^\infty x L_1(x)dx = o(t^{-1}).$$

Since

$$\left| t^2 \int_t^\infty L_1(x)dx \right| \le \int_t^\infty x^2 |L_1(x)| dx = o(1), \qquad t \to \infty,$$

then

$$\int_t^\infty L_1(x)dx = o(t^{-2}), \qquad t \to \infty.$$

Denote the Laplace transform of the expression $t\int_{x=0}^t L_1^{*(2)}(t-x)dL_1(x)$ by $\hat{L}(\alpha)$. Then

$$\hat{L}(\alpha) = \int_{t=0}^\infty e^{-\alpha t} t \int_{x=0}^t L_1^{*(2)}(t-x)dL_1(x)\, dt =$$

$$= \int_{x=0}^\infty \int_{t=x}^\infty e^{-\alpha t} t L_1^{*(2)}(t-x)dt\, dL_1(x) = \int_{x=0}^\infty e^{-\alpha x} \int_{t=0}^\infty e^{-\alpha t}(t+x) L_1^{*(2)}(t)dt\, dL_1(x) =$$

$$= \int_{x=0}^\infty e^{-\alpha x} dL_1(x) \int_{t=0}^\infty t e^{-\alpha t} L_1^{*(2)}(t)dt + \int_{x=0}^\infty x e^{-\alpha x} dL_1(x) \int_{t=0}^\infty e^{-\alpha t} L_1^{*(2)}(t)dt.$$

Therefore

$$\lim_{\alpha\to 0}\alpha\hat{L}(\alpha) = 0\cdot\left(b_1 l_{1,1}^{*(2)} + l_{1,0}l_{1,0}^{*(2)}\right) = 0.$$

So, by the Tauberian theorems

$$\lim_{t\to\infty} t\int_0^t L_1^{*(2)}(t-x)dL_1(x) = 0,$$

thus

$$\int_0^t L_1^{*(2)}(t-x)dL_1(x) = o(t^{-1}),\qquad t\to\infty.$$

Finally, it means that

$$L_1^{*(3)}(t) = o(t^{-1}),\qquad t\to\infty.$$

This completes the proof.

Let $l_{1,0}^{*(3)} = \int_0^\infty L_1^{*(3)}(t)dt$ whenever $\int_0^\infty\left|L_1^{*(3)}(t)\right|dt < \infty$.

**Lemma 2.8.** Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_1$, $\mu_5$, $n_{4,1}$ exist. Then

$$l_{1,0}^{*(3)} = -a_1^{*(2)}l_{1,2} - 2b_1^{*(2)}l_{1,1} - a_1 l_{1,1}^{*(2)} + b_1 l_{1,0}^{*(2)}.$$

**Proof.** From Lemma 2.7

$$L_1^{*(3)}(t) = -a_1^{*(2)}t^2L_1(t) - b_1^{*(2)}tL_1(t) - a_1\int_t^\infty L_1^{*(2)}(x)dx - \left(2a_1^{*(2)}t + b_1^{*(2)}\right)\int_t^\infty L_1(x)dx +$$

$$+2a_1^{*(2)}\int_t^\infty xL_1(x)dx + \int_0^t L_1^{*(2)}(t-x)dL_1(x).$$

Since $\int_0^\infty t^s|L_1(t)|dt < \infty,\ s = 0, 1, 2,$ then

$$\int_{t=0}^\infty\left|\int_{x=t}^\infty L_1^{*(2)}(x)dx\right|dt \le \int_{x=0}^\infty\left|L_1^{*(2)}(x)\right|\int_{t=0}^x dt\,dx = \int_{x=0}^\infty x\left|L_1^{*(2)}(x)\right|dx < \infty,$$

$$\int_{t=0}^\infty\left|\int_{x=t}^\infty L_1(x)dx\right|dt \le \int_{x=0}^\infty|L_1(x)|\int_{t=0}^x dt\,dx = \int_{x=0}^\infty x|L_1(x)|dx < \infty,$$

$$\int_{t=0}^\infty\left|t\int_{x=t}^\infty L_1(x)dx\right|dt \le \int_{x=0}^\infty|L_1(x)|\int_{t=0}^x tdt\,dx = \frac{1}{2}\int_{x=0}^\infty x^2|L_1(x)|dx < \infty,$$

$$\int_{t=0}^\infty\left|\int_{x=t}^\infty xL_1(x)dx\right|dt \le \int_{x=0}^\infty x|L_1(x)|\int_{t=0}^x dt\,dx = \int_{x=0}^\infty x^2|L_1(x)|dx < \infty.$$

On the other hand,

$$\int_{t=0}^\infty\left|\int_0^t L_1^{*(2)}(t-x)dL_1(x)\right|dt \le \int_{x=0}^\infty\int_{t=x}^\infty\left|L_1^{*(2)}(t-x)\right|dt\,dL_1(x) = \int_{x=0}^\infty|dL_1(x)|\int_{t=0}^\infty\left|L_1^{*(2)}(t)\right|dt$$

$$< \infty.$$

$L_1^{*(3)}(t)$ is integrable and

$$l_{1,0}^{*(3)} = -a_1^{*(2)}l_{1,2} - b_1^{*(2)}l_{1,1} - a_1\int_0^\infty xL_1^{*(2)}(x)dx - a_1^{*(2)}\int_x^\infty x^2L_1(x)dx -$$

$$-b_1^{*(2)}\int_x^\infty xL_1(x)dx + a_1^{*(2)}\int_0^\infty x^2L_1(x)dx + \int_{x=0}^\infty dL_1(x)\int_{t=0}^\infty L_1^{*(2)}(t)dt =$$

$$= -a_1^{*(2)}l_{1,2} - b_1^{*(2)}l_{1,1} - a_1 l_{1,1}^{*(2)} - a_1^{*(2)}l_{1,2} - b_1^{*(2)}l_{1,1} + a_1^{*(2)}l_{1,2} + b_1 l_{1,0}^{*(2)} =$$

$$= -a_1^{*(2)}l_{1,2} - 2b_1^{*(2)}l_{1,1} - a_1 l_{1,1}^{*(2)} + b_1 l_{1,0}^{*(2)}.$$

This completes the proof.

**Lemma 2.9.** Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_1$, $\mu_5$, $n_{4,1}$ exist. Then

$$M_1^{*(4)}(t) = a_1^{*(4)}t^4 + b_1^{*(4)}t^3 + c_1^{*(4)}t^2 + d_1^{*(4)}t + e_1^{*(4)} + L_1^{*(4)}(t),$$

where $a_1^{*(4)} = \frac{1}{4}a_1a_1^{*(3)}$; $b_1^{*(4)} = \frac{1}{3}a_1b_1^{*(3)} + a_1^{*(3)}b_1$; $c_1^{*(4)} = \frac{1}{2}a_1c_1^{*(3)} + b_1b_1^{*(3)} + 3a_1^{*(3)}l_{1,0}$;

$$d_1^{*(4)} = a_1d_1^{*(3)} + b_1c_1^{*(3)} + 2b_1^{*(3)}l_{1,0} - 6a_1^{*(3)}l_{1,1};$$

$$e_1^{*(4)} = a_1l_1^{*(3)} + d_1^{*(3)}b_1 + c_1^{*(3)}l_{1,0} - 2b_1^{*(3)}l_{1,1} + 3a_1^{*(3)}l_{1,2};$$

$$L_1^{*(4)}(t) = o(1), \qquad t \to \infty.$$

**Proof.** Since

$$M_1^{*(4)}(t) = \int_0^t \left(M_1^{*(3)}(t-x)\right) dM_1(x),$$

by using Theorem 2.1 of [2, pp.143-144] and Lemma 2.7, we can write

$$M_1^{*(4)}(t) = a_1^{*(4)}t^4 + b_1^{*(4)}t^3 + c_1^{*(4)}t^2 + d_1^{*(4)}t + e_1^{*(4)} + L_1^{*(4)}(t),$$

where $a_1^{*(4)} = \frac{1}{4}a_1a_1^{*(3)}$; $b_1^{*(4)} = \frac{1}{3}a_1b_1^{*(3)} + a_1^{*(3)}b_1$; $c_1^{*(4)} = \frac{1}{2}a_1c_1^{*(3)} + b_1b_1^{*(3)} + 3a_1^{*(3)}l_{1,0}$;

$$d_1^{*(4)} = a_1d_1^{*(3)} + b_1c_1^{*(3)} + 2b_1^{*(3)}l_{1,0} - 6a_1^{*(3)}l_{1,1};$$

$$e_1^{*(4)} = a_1l_1^{*(3)} + b_1d_1^{*(3)} + c_1^{*(3)}l_{1,0} - 2b_1^{*(3)}l_{1,1} + 3a_1^{*(3)}l_{1,2}.$$

Let’s compute the $L_1^{*(4)}(t)$:

$$L_1^{*(4)}(t) = -a_1\int_t^\infty L_1^{*(3)}(x)dx + \left(a_1^{*(3)}t^3 + b_1^{*(3)}t^2 + c_1^{*(3)}t + d_1^{*(3)}\right)L_1(t) -$$

$$-\left(3a_1^{*(3)}t^2 + 2b_1^{*(3)}t + c_1^{*(3)}\right)\left(tL_1(t) + \int_t^\infty L_1(x)dx\right) +$$

$$+\left(3a_1^{*(3)}t + b_1^{*(3)}\right)\left(t^2L_1(t) + 2\int_t^\infty xL_1(x)dx\right) - a_1^{*(3)}\left(t^3L_1(t) + 3\int_t^\infty x^2L_1(x)dx\right) +$$

$$+\int_0^t L_1^{*(3)}(t-x)dL_1(x).$$

Since $L_1(t) = o(t^{-3})$ as $t \to \infty$, then

$$tL_1(t) = o(t^{-2}), \qquad t^2L_1(t) = o(t^{-1}), \qquad t^3L_1(t) = o(1), \qquad t \to \infty.$$

Since

$$\int_0^\infty \left|L_1^{*(3)}(x)\right|dx < \infty, \qquad \int_0^\infty x^2|L_1(x)|dx < \infty,$$

As $t \to \infty$, we obtain that

$$\int_t^\infty L_1^{*(3)}(x)dx = o(1), \qquad \int_t^\infty L_1(x)dx = o(t^{-2}), \quad t \to \infty,$$

$$\int_t^\infty xL_1(x)dx = o(t^{-1}), \qquad \int_t^\infty x^2 L_1(x)dx = o(1), \qquad t \to \infty.$$

Denote the Laplace transform of the expression $\int_{x=0}^{t} L_1^{*(3)}(t-x)dL_1(x)$ by $\hat{L}(\alpha)$. Then

$$\hat{L}(\alpha) = \int_{t=0}^{\infty} e^{-\alpha t} \int_{x=0}^{t} L_1^{*(3)}(t-x)dL_1(x)\,dt =$$

$$= \int_{x=0}^{\infty} \int_{t=x}^{\infty} e^{-\alpha t} L_1^{*(3)}(t-x)dt\,dL_1(x) = \int_0^\infty e^{-\alpha x} dL_1(x) \int_0^\infty e^{-\alpha t} L_1^{*(3)}(t)dt.$$

Therefore

$$\lim_{\alpha \to 0} \alpha \hat{L}(\alpha) = 0.$$

So, by the Tauberian theorems

$$\lim_{t\to\infty} \int_0^t L_1^{*(3)}(t-x)dL_1(x) = 0,$$

thus

$$\int_0^t L_1^{*(3)}(t-x)dL_1(x) = o(1), \qquad t \to \infty.$$

Finally, it means that

$$L_1^{*(3)}(t) = o(1), \qquad t \to \infty.$$

This completes the proof.

**Lemma 2.10.** If Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_1$, $\mu_5$, $n_{4,1}$ exist. Then

$$M_2 * M_1^{*(2)}(t) = a_1^{*(3)} t^3 + b_1^{*(3)} t^2 + c_1^{*(3)} t + d_1^{*(3)} + L_1^{*(3)}(t),$$

where $L_1^{*(3)} = o(t^{-1}),\ t \to \infty,\ a_1^{*(3)} = \frac{1}{3} a_1 a_1^{*(2)},\ b_1^{*(3)} = \frac{1}{2} a_1 b_1^{*(2)} + a_1^{*(2)} b_1,$

$$c_1^{*(3)} = a_1 c_1^{*(2)} + b_1 b_1^{*(2)} + 2a_1^{*(2)} l_{1,0}, \qquad d_1^{*(3)} = a_1 l_1^{*(2)} + b_1 c_1^{*(2)} + b_1^{*(2)} l_{1,0} - 2a_1^{*(2)} l_{1,1}.$$

**Theorem 1.** Let $F(x)$ be a strongly non-lattice distribution function and $\lambda_4$, $\mu_5$, $n_{4,4}$ exist. Then

$$D_4(t) = D_{4,4} t^4 + D_{4,3} t^3 + D_{3,2} t^2 + D_{3,1} t + D_{3,0} + L_{D_3}(t),$$

where $D_{4,4} = 24a_1^{*(4)}; D_{4,3} = 24b_1^{*(4)} + 36a_1^{*(3)};\ D_{4,2} = 24c_1^{*(4)} + 36b_1^{*(3)} + 6a_2^{*(2)} + 8a_{1*3};$
$D_{4,1} = 24d_1^{*(4)} + 36c_1^{*(3)} + 6b_2^{*(2)} + 8b_{1*3} + a_4; D_{4,0} = 24e_1^{*(4)} + 36d_1^{*(3)} + 6c_2^{*(2)} + 8c_{1*3} + b_4;$

$$L_{D_4}(t) = o(1), \qquad t \to \infty.$$

**Proof**. From Lemma 2.2 we can write

$$D_4(t) = M_4(t) + 8M_3 * M_1(t) + 6M_2^{*(2)}(t) + 36M_2 * M_1^{*(2)}(t) + 24M_1^{*(4)}(t).$$

From Corollary 1 we can write as $t \to \infty$

$$M_1 * M_3(t) = a_{1*3} t^2 + b_{1*3} t + c_{1*3} + L_{1*3}(t), \qquad L_{1*3}(t) = o(t^{-2}).$$

From Corollary 2 we can write as $t \to \infty$

$$M_2^{*(2)}(t) = a_2^{*(2)}t^2 + b_2^{*(2)}t + c_2^{*(2)} + L_2^{*(2)}(t), \qquad L_2^{*(2)}(t) = o(t^{-2}).$$

From Lemma 2.7 we can write

$$M_1^{*(3)}(t) = a_1^{*(3)}t^3 + b_1^{*(3)}t^2 + c_1^{*(3)}t + d_1^{*(3)} + L_1^{*(3)}(t),$$

$$M_1^{*(4)}(t) = a_1^{*(4)}t^4 + b_1^{*(4)}t^3 + c_1^{*(4)}t^2 + d_1^{*(4)}t + e_1^{*(4)} + L_1^{*(4)}(t).$$

From the Theorem 2.1 of [2, pp. 143], we can write

$$M_4(t) = a_4t + b_4 + L_4(t).$$

By combining this results we can write

$$D_4(t) = D_{4,4}t^4 + D_{3,2}t^2 + D_{3,1}t + D_{3,0} + L_{D_3}(t),$$

where $D_{4,4} = 24a_1^{*(4)}; D_{4,3} = 24b_1^{*(4)} + 36a_1^{*(3)}; D_{4,2} = 24c_1^{*(4)} + 36b_1^{*(3)} + 6a_2^{*(2)} + 8a_{1*3};$
$D_{4,1} = 24d_1^{*(4)} + 36c_1^{*(3)} + 6b_2^{*(2)} + 8b_{1*3} + a_4;\ D_{4,0} = 24e_1^{*(4)} + 36d_1^{*(3)} + 6c_2^{*(2)} + 8c_{1*3} + b_4$

$$L_{D_4}(t) = 24L_1^{*(4)}(t) + 36L_1^{*(3)}(t) + 6L_2^{*(2)}(t) + 8L_{1*2}(t) + L_4(t) =$$
$$= o(1) + o(t^{-1}) + o(t^{-2}) + o(t^{-2}) + o(t^{-2}) = o(1), \qquad t \to \infty.$$

This completes the proof.

## 4. Conclusion

In this work, the study of stochastic processes with dependent random variables is quite difficult. In the presented article, precisely such a process has been examined. It should also be noted that such stochastic prosses arise in fields like stochastic control theory, insurance and reliability theories, analyzing systems and processes in areas like operations research, economics, finance, and engineering. The results obtained in the article can be applied in the listed application fields.

## References

[1] R.T. Aliyev, T.A.Khanıyev, On the moments of a semi-Markovian random walk with Gaussian distribution of summands, Comm. Statist. Theory Methods. 43 (2014) pp.90-104.

[2] R.T. Aliyev, V. Bayramov, On the asymptotic behaviour of the covariance function of the rewards of a multivariate renewal-reward process, Statistics and Probability Letters. 127 (2017) pp.138-149.

[3] M. Brown, H.A.Solomon, Second-order approximation for the variance of a renewal reward process, Stochastic Process. Appl. 3 (1975) pp.301-314.

[4] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 2, second ed. John Wiley and Sons, New York. 1971.

[5] T.A. Khaniyev, T. Kesemen, R.T. Aliyev, A. Kokangul, Asymptotic expansions for the moments of a semi-Markovian random walk with exponential distributed interference of chance, Statist. Probab. Lett. 78 No.6 (2008) pp.785-793.

[6] B. Patch, Y. Nazarathy, T. Taimire, A correction term for the covariance of renewal-reward processes with multivariate rewards, Statist. Probab. Lett. 102 (2015) pp.1-7.

[7] S.M. Ross, Applied Probability Models with Optimization Applications, Holden-Day, San Francisco, Calif. (1970).

[8] S.M. Ross, Stochastic Processes, second ed. John Wiley and Sons, New York. Smith, W.L., 1955. Regenerative stochastic processes. Proc.Roy.Soc. 232 (1996) pp.6-31.

[9] W.L. Smith, On the cumulants on renewal processes, Biometrika. 46 (1959) pp.1-29.