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Necessary optimality conditions of quasi-singular controls for a continuous stochastic Rosser-type control problem

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ABSTRACT

Necessary first- and second-order optimality conditions are obtained for one stochastic optimal control problem described by a stochastic system of first-order nonlinear hyperbolic equations; these conditions are the stochastic analogs of the linearized maximum principle and the optimality conditions of quasi-singular controls, respectively. Such control problems are encountered in the optimization of certain chemical-technological processes under the influence of random effects.

1. Introduction

Issues of obtaining necessary first- and second-order optimality conditions for optimal control problems of deterministic dynamical systems described by first-order hyperbolic equations have been addressed in [1-3] and others.

A similar problem of optimal control in the stochastic case is examined in [4, 5], where necessary first-order optimality conditions (an analog of Pontryagin's maximum principle, linearized maximum principle, an analog of Euler's equation [6, p.167; 7, p.30]) are established.

The relevance of research in this area is dictated by the need for the most accurate description of, for example, automatic control systems, a number of chemical-technological processes [8; 9, p.26], a realistic version of which is a stochastic description that takes into account the impact of random noise on the control object.

In this study, we use a modified version of the increment method to derive the formula for the second-order increment of the quality criterion for the quality functional, which allows us to obtain the necessary first-order optimality conditions of the linearized Pontryagin maximum principle type, and to investigate quasi-singular controls (i.e., the case of degeneration of the first-order optimality condition) in the considered stochastic problem described by a system of first-order stochastic nonlinear hyperbolic equations written in the canonical form.

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2. Problem statement

Suppose that the controlled process in the specified domain $D = [t_0, t_1] \times [x_0, x_1]$ is described by the following system of stochastic nonlinear Rosser-type differential equations [10]

$$\frac{\partial z(t,x)}{\partial t} = f(t,x,z,y,u) + p(t,x,z)\frac{\partial W_1(t,x)}{\partial t},$$
$$\frac{\partial y(t,x)}{\partial x} = g(t,x,z,y,u) + q(t,x,y)\frac{\partial W_2(t,x)}{\partial x}, (t,x) \in D, \quad (1)$$

with Goursat-type boundary conditions

$$z(t_0, x) = a(x), x \in [x_0, x_1], y(t, x_0) = b(t), t \in [t_0, t_1]. \quad (2)$$

Here, $\big(z(t,x), y(t,x)\big)$ is a $(n+m)$-dimensional required vector function; $f(t,x,z,y,u)$ and $g(t,x,z,y,u)$ are n- and m-dimensional vector function, respectively, that is continuous, in the set of variables together with partial derivatives (z,y), up to the second order, it is also assumed that there exist continuous derivatives $f_{zu}, f_{yu}, f_{uu}, g_{zu}, g_{yu}, g_{uu}$; $p(t,x,z)\big(q(t,x,y)\big)$ is a $(n \times n)$ $\big((m \times m)\big)$-dimensional matrix function that is continuous in the set of variables together with partial derivatives $z(y)$; vector functions $\frac{\partial W_1(t,x)}{\partial t}, \frac{\partial W_2(t,x)}{\partial x}$ are derivatives, in t and x, respectively of the two-parameter Wiener process $W_1(t,x)$, $W_2(t,x)$ [11, 12], and $a(x), b(t)$ are specified vector-functions of corresponding dimensions, measurable and bounded on $[x_0, x_1]$, $[t_0, t_1]$, respectively.

A class of r-dimensional vector functions $u(t,x)$ that are measurable with respect to a non-decreasing Borel σ-algebra $\mathcal{F} = \bar{\sigma}(W(\tau, s), t_0 \le \tau \le t, x_0 \le s \le x)$ and bounded on D with values from a specified nonempty, bounded and convex set $U \subset R^r\big(u(t,x) \in L_\infty(D, U)\big)$ is taken as admissible controls.

Linear stochastic differential equations like (1)-(2) were studied in [13], so the solution of system (1)-(2) corresponding to a certain admissible control $u(t,x)$ is understood in the indicated sense.

It is always assumed that for each admissible control, the corresponding solution of system (1)-(2) exists and is unique in D.

Let us consider the problem of the minimum of the functional

$$S(u) = E\left\{\int_{t_0}^{t_1}\int_{x_0}^{x_1} F_3\big(t,x,z(t,x),y(t,x),u(t,x)\big)dxdt + \right.$$
$$\left. + \int_{x_0}^{x_1} F_1\big(x, z(t_1,x)\big)dx + \int_{t_0}^{t_1} F_2\big(t, y(t,x_1)\big)dt \right\} \quad (3)$$

determined on the solutions of boundary value problem (1)-(2) generated by all possible admissible controls.

Here, $F_1(x,z), F_2(t,y), F_3(t,x,z,y,u)$ are specified scalar functions, continuous in the set of variables together with partial derivatives on the state vector up to the second order, and there exist continuous derivatives $\frac{\partial^2 F_3}{\partial z \partial u}, \frac{\partial^2 F_3}{\partial y \partial u}$. E is the sign of mathematical expectation.

Our goal is to derive the stochastic analogue of linearized maximum principle [14] and the necessary second-order optimality conditions for quasi-singular controls in the considered stochastic control problem with distributed parameters (1)-(3).

3. Formula for the second-order increment of the quality criterion

Suppose $\big(u(t,x), z(t,x), y(t,x)\big)$ is a fixed, and $\big(\bar{u}(t,x) = u(t,x) + u(t,x), \bar{z}(t,x) = z(t,x) + \Delta z(t,x), \bar{y}(t,x) = y(t,x) + \Delta y(t,x)\big)$ are arbitrary admissible processes.

Let us introduce an analog of the Hamilton-Pontryagin function

$$H(t,x,z,y,u,\psi,\lambda) = -F_3(t,x,z,y,u) + \psi' f(t,x,z,y,u) + \lambda' g(t,x,z,y,u)$$

and notations:

$$H_z[t,x] = H_z\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),\lambda(t,x)\big),$$

the notations $H_{zz}[t,x], H_{zy}[t,x], H_{zu}[t,x], H_{uu}[t,x], H_{yy}[t,x]$ have a similar meaning.
Here,$(\psi(t,x), \lambda(t,x), \alpha(t,x), \beta(t,x)) \in L_\infty(D,R^n) \times L_\infty(D,R^m) \times L_\infty(D,R^{n\times n}) \times L_\infty(D,R^{m\times m})$ are solutions of the following stochastic conjugate problem:

$$\psi_t(t,x) = -\frac{\partial H(t,x,z,y,u,\psi,\lambda)}{\partial z} + \alpha(t,x)\frac{\partial W(t,x)}{\partial t}, \qquad \psi(t_1,x) = \frac{\partial F_1\big(x,z(t_1,x)\big)}{\partial z},$$

$$\lambda_x(t,x) = -\frac{\partial H(t,x,z,y,u,\psi,\lambda)}{\partial y} + \beta(t,x)\frac{\partial W(t,s)}{\partial s}, \qquad \lambda(t,x_1) = \frac{\partial F_2\big(t,y(t,x_1)\big)}{\partial y}.$$

Applying Taylor's formula to the expressions $F_1\big(x,\bar{z}(t_1,x)\big) - F_1\big(x,z(t_1,x)\big)$, $F_2\big(t,\bar{y}(t,x_1)\big) - F_2\big(t,y(t,x_1)\big), H(t,x,\bar{z},\bar{y},\bar{u},\psi,\lambda) - H(t,x,z,y,u,\psi,\lambda)$, and taking into account the introduced notations, we obtain

$$\Delta S(u) = S\big(\bar{u}(t,x)\big) - S\big(u(t,x)\big) = E\Bigg\{-\int_{t_0}^{t_1}\int_{x_0}^{x_1} H_u'[t,x]\Delta u(t,x)\,dxdt +$$
$$+\frac{1}{2}\int_{x_0}^{x_1} \Delta z'(t_1,x)\frac{\partial^2 F_1\big(x,z(t_1,x)\big)}{\partial z^2}\Delta z(t_1,x)\,dx +$$
$$+\frac{1}{2}\int_{t_0}^{t_1} \Delta y'(t,x_1)\frac{\partial^2 F_2\big(t,y(t,x_1)\big)}{\partial y^2}\Delta y(t,x_1)\,dt - \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta z(t,x) H_{zz}[t,x]\Delta z(t,x)\,dxdt -$$
$$-\frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta z'(t,x) H_{zy}[t,x]\Delta y(t,x)\,dxdt - \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta y'(t,x) H_{yz}[t,x]\Delta z(t,x)\,dxdt -$$
$$-\frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta y'(t,x) H_{yy}[t,x]\Delta y(t,x)\,dxdt - \int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta u'(t,x) H_{uz}[t,x]\Delta z(t,x)\,dxdt -$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta u'(t,x) H_{uy}[t,x]\Delta y(t,x)\,dxdt - \int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta u'(t,x) H_{uu}[t,x]\Delta u(t,x)\,dxdt\Bigg\} +$$
$$+\eta_1(t,x;\Delta u), \tag{4}$$

where by definition

$$\eta_1(t,x;\Delta u) = E\left\{\int_{x_0}^{x_1} o_1(\|\Delta z(t_1,x)\|^2)dx + \int_{t_0}^{t_1} o_2(\|\Delta y(t,x_1)\|^2)dt - \right.$$

$$\left. - \int_{t_0}^{t_1}\int_{x_0}^{x_1} o_3([\![\|\Delta z(t,x)\| + \|\Delta y(t,x)\| + \|\Delta u(t,x)\|]\!]^2)dxdt\right\}. \quad (5)$$

It can be shown that for almost all $(t,x) \in D$, the following estimates are valid:

$$\|\Delta z(t,x)\| \le K_1 E\left(\int_{t_0}^{t}\|\Delta u(\tau,x)\|d\tau + \int_{t_0}^{t}\int_{x_0}^{x}\|\Delta u(\tau,s)\|dsd\tau\right), \quad (6)$$

$$\|\Delta y(t,x)\| \le K_2 E\left(\int_{x_0}^{x}\|\Delta u(t,s)\|ds + \int_{t_0}^{t}\int_{x_0}^{x}\|\Delta u(\tau,s)\|dsd\tau\right), \quad (7)$$

where $K_i = const > 0, i = 1,2$.

We will determine the special increment of the admissible control $u(t,x)$ using the formula:

$$\Delta u_\varepsilon(t,x) = \varepsilon\big(v(t,x) - u(t,x)\big), (t,x) \in D, \quad (8)$$

where $\varepsilon \in [0,1]$, and $v(t,x) \in L_\infty(D,U)$is an arbitrary measurable and D-bounded r-dimensional vector functions. We took advantage of the fact that the control domain U is a convex set.

Here, $\big(\delta z(t,x), \delta y(t,x)\big)$ is a variation of the state vector, which is the solution of the stochastic equation in variations

$$(\delta z)_t = f_z[t,x]\delta z + f_y[t,x]\delta y + f_u[t,x]\delta u + p_z[t,x]\delta z\frac{\partial W_1(t,x)}{\partial t},$$

$$(\delta y)_x = g_z[t,x]\delta z + g_y[t,x]\delta y + g_u[t,x]\delta u + q_y[t,x]\delta y\frac{\partial W_2(t,x)}{\partial x}, \quad (9)$$

with zero boundary condition

$$\delta z(t_0,x) = 0, x \in [x_0,x_1], \delta y(t,x_0) = 0, t \in [t_0,t_1]. \quad (10)$$

We denote by $(\Delta z_\varepsilon(t,x),\ y_\varepsilon(t,x))$ the special increment of system (1)-(2), corresponding to special increment (8) of the control.

Using estimates (6), (7) and formula (8) according to the scheme similar to [15, 16] we obtain the expansions

$$\Delta z_\varepsilon(t,x) = \varepsilon\delta z(t,x) + o(\varepsilon;t,x),$$
$$\Delta y_\varepsilon(t,x) = \varepsilon\delta y(t,x) + o(\varepsilon;t,x).$$

Using these expansions and taking into account (6)-(10), we obtain the following formula for the increment of the quality criterion:

$$S\big(\bar{u}(t,x)\big) - S\big(u(t,x)\big) = E\left\{-\varepsilon\int_{t_0}^{t_1}\int_{x_0}^{x_1} H_u'[t,x](v(t,x) - u(t,x))dxdt + \right.$$

$$+\frac{1}{2}\varepsilon^2\left\{\int_{x_0}^{x_1}\delta z'(t_1,x)\frac{\partial^2 F_1\big(x,z(t_1,x)\big)}{\partial z^2}\delta z(t_1,x)\,dx - \right.$$

$$-\int_{t_0}^{t_1}\delta y'(t,x_1)\frac{\partial^2 F_2\big(t,y(t,x_1)\big)}{\partial y^2}\delta y(t,x_1)\,dt - \int_{t_0}^{t_1}\int_{x_0}^{x_1}\delta z'(t,x)H_{zz}[t,x]\delta z(t,x)\,dxdt -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\delta z'(t,x)H_{zy}[t,x]\delta y(t,x)\,dxdt-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\delta y'(t,x)H_{yz}[t,x]\delta z(t,x)\,dxdt-$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\delta y'(t,x)H_{yy}[t,x]\delta y(t,x)\,dxdt-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\delta u'(t,x)H_{uz}[t,x]\delta z(t,x)\,dxdt-$$
$$\left.-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\delta u'(t,x)H_{uy}[t,x]\delta y(t,x)\,dxdt-\int_{t_0}^{t_1}\int_{x_0}^{x_1}\delta u'(t,x)H_{uu}[t,x]\delta u(t,x)\,dxdt\right\}+$$
$$+o(\varepsilon^2), \qquad (11)$$

Note that the last formula will be of key importance for obtaining an analog of the linearized maximum principle and studying the quasi-singular control of the problem under consideration.

4. Necessary first- and second-order optimality conditions

From formula (11), due to the arbitrariness of $\varepsilon \in [0,1]$, an analogue of the linearized maximum principle follows for stochastic problem (1)-(3) under consideration.

Let us formulate the obtained result.

Theorem 1. The optimality of the admissible control $u(t,x)$ requires that the relation

$$EH_u'\big(\theta,\xi,z(\theta,\xi),y(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi),\xi(\theta,\xi)\big)(v-u(\theta,\xi))\le 0, \qquad (12)$$

hold for all $(\theta,\mu)\in[t_0,t_1)\times[x_0,x_1)$.

Here and in the following $(\theta,\xi)\in[t_0,t_1)\times[x_0,x_1)$ is an arbitrary Lebesgue point (proper point [1, 2] of the control $u(t,x)$), $v\in U$.

Definition [6, 7]. If along the admissible process $(u(t,x),z(t,x),y(t,x))$ for all $v\in R^r$, $(\theta,\xi)\in[t_0,t_1)\times[x_0,x_1)$

$$EH_u'\big(\theta,\xi,z(\theta,\xi),y(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi),\xi(\theta,\xi)\big)\big(v-u(\theta,\xi)\big)=0, \qquad (13)$$

then the control $u(t,x)$ will be called a quasi-singular control.

The following statement is true.

Theorem 2. The optimality of the quasi-singular control $u(t,x)$ requires that the relation

$$E\big(v-u(\theta,\xi)\big)'H_{uu}[\theta,\xi]\big(v-u(\theta,\xi)\big)\le 0, \qquad (14)$$

hold for all $(\theta,\mu)\in[t_0,t_1)\times[x_0,x_1)$, and $v\in U$.

Inequality (14), as well as condition (12), is a direct consequence of the Pontryagin maximum principle for the stochastic problem under consideration and, therefore, carries, in general, limited information about the quasi-singular control suspicious for optimality. On the other hand, the degeneracy of optimality condition (14) is not impossible. This, in turn, dictates to obtain new, more informative, necessary optimality conditions for quasi-singular controls.

Interpreting the equations in variations (9), (10) as linear inhomogeneous stochastic equations in $\delta z(t,x),\delta y(t,x)$, based on the analog of the Cauchy formula from [13], we obtain

$$\delta z(t,x)=\int_{t_0}^{t}V_{11}(t,x;\tau,x)f_u[\tau,x]\,\delta u(\tau,x)d\tau+\int_{t_0}^{t}\int_{x_0}^{x}A(t,x;\tau,s)\delta u(\tau,s)dsd\tau, \qquad (15)$$

$$\delta y(t,x) = \int_{x_0}^{x} V_{22}(t,x;t,s)g_u[t,s]\,\delta u(t,s)ds + \int_{t_0}^{t}\int_{x_0}^{x} B(t,x;\tau,s)\delta u(\tau,s)dsd\tau, \quad (16)$$

where by definition

$$A(t,x;\tau,s) = \frac{\partial V_{11}(t,x;\tau,s)}{\partial x} f_u[\tau,s] + \frac{\partial V_{12}(t,x;\tau,s)}{\partial x} g_u[\tau,s],$$

$$B(t,x;\tau,s) = \frac{\partial V_{21}(t,x;\tau,s)}{\partial t} f_u[\tau,s] + \frac{\partial V_{22}(t,x;\tau,s)}{\partial t} g_u[\tau,s].$$

Here, $V_{ij}(t,x;\tau,s), (t_0 \le \tau \le t \le t_1, x_0 \le s \le x \le x_1), i,j = 1,2$ are matrix functions that are solutions of the following stochastic problems [14]:

$$\frac{\partial V_{11}(t,x;\tau,s)}{\partial \tau} = -V_{11}(t,x;\tau,s)f_z[\tau,s] - V_{12}(t,x;\tau,s)g_z[\tau,s] - $$
$$-V_{11}(t,x;\tau,s)p[\tau,s]\frac{\partial W(\tau,s)}{\partial \tau},$$

$$\frac{\partial V_{12}(t,x;\tau,s)}{\partial s} = V_{11}(t,x;\tau,s)f_y[\tau,s] - V_{12}(t,x;\tau,s)g_y[\tau,s] - $$
$$-V_{12}(t,x;\tau,s)q[\tau,s]\frac{\partial W(\tau,s)}{\partial s},$$

$$\frac{\partial V_{21}(t,x;\tau,s)}{\partial \tau} = -V_{21}(t,x;\tau,s)f_z[\tau,s] - V_{22}(t,x;\tau,s)g_z[\tau,s] - $$
$$-V_{21}(t,x;\tau,s)p[\tau,s]\frac{\partial W(\tau,s)}{\partial \tau},$$

$$\frac{\partial V_{22}(t,x;\tau,s)}{\partial s} = -V_{21}(t,x;\tau,s)f_z[\tau,s] - V_{21}(t,x;\tau,s)g_z[\tau,s]$$
$$-V_{22}(t,x;\tau,s)q[\tau,s]\frac{\partial W(\tau,s)}{\partial s},$$

$$V_{11}(t,x;t,s) = E_1, V_{12}(t,x;\tau,x) = 0, t_0 \le \tau \le t, V_{21}(t,x;t,s) = 0,$$

$V_{22}(t,x;t,s) = E_2,\ x_0 \le s \le x,$ where E_1, E_2 are identity matrices of corresponding dimensions.

For further discussion we will introduce into consideration the $(n \times n)$ −matrix function $R(x,\tau,s)$ and $(m \times m)$ −matrix function $Q(t,\tau,s)$ using the following formulas

$$R(x,\tau,s) = \int_{\max(\tau,s)}^{t_1} V_{11}'(t,x;\tau,x)\,H_{zz}[t,x]V_{11}(t,x;s,x)dt - $$
$$-V_{11}'(t_1,x;\tau,x)\frac{\partial^2 F_1\big(x,z(t_1,x)\big)}{\partial z^2}V_{11}(t_1,x;s,x),$$

$$Q(t,\tau,s) = \int_{\max(\tau,s)}^{x_1} V_{22}'(t,x;t,x)\,H_{yy}[t,x]V_{22}(t,x;t,s)dx - $$
$$-V_{22}'(t,x_1;t,\tau)\frac{\partial^2 F_2\big(t,y(t,x_1)\big)}{\partial y^2}V_{22}(t,x_1;t,s).$$

Considering $u(t,x)$ a quasi-singular control, its special increment is constructed by the formula

$$\Delta u(t,x;\mu) = \mu\delta u(t,x), (t,x) \in D. \quad (17)$$

Here, $0 \le \mu \le 1$, and $\delta u(t,x) \in U$ is a vector function such that the variated control $\bar{u}_\mu(t,x) = u(t,x) + \mu\delta u(t,x)$ is admissible.

Now, considering that $u(t,x)$ is an optimal control, its special variation is constructed by the formula

$$\delta u_\varepsilon(t,x) = \begin{cases} l_1(t) - u(t,x), (t,x) \in D_\varepsilon = [t_0,t_1] \times [\xi, \xi+\varepsilon), \\ 0, (t,x) \in D/D_\varepsilon \end{cases} \tag{18}$$

where $\varepsilon > 0$ is a sufficiently small number, $\xi \in [x_0,x_1)$, and $l_1(t) \in KC_r([t_0,t_1],R^r)$.

We denote by $(\delta z_\varepsilon(t,x), \delta y_\varepsilon(t,x))$ a solution of the stochastic equation in variations (9)-(10) corresponding to variation (18) of the control $u(t,x)$.

Then it follows from representations (15), (16) that for almost all (t,x)

$$\delta z_\varepsilon(t,x) \sim \begin{cases} \varepsilon^0, (t,x) \in [t_0,t_1] \times [\xi,\xi+\varepsilon), \\ \varepsilon, (t,x) \in [t_0,t_1] \times [\xi+\varepsilon, x_1], \end{cases} \tag{19}$$

$$\delta y_\varepsilon(t,x) \sim \varepsilon, (t,x) \in [t_0,t_1] \times [\xi, x_1]. \tag{20}$$

Hence, following the works [1, 2], by means of representations (15), (16) of the solutions δz and δy, after simple transformations we obtain:

$$S\big(u(t,x) + \mu\delta u_\varepsilon(t,x)\big) - S\big(u(t,x)\big) == -0{,}5\mu^2 E\left\{\varepsilon\left[\int_{t_0}^{t_1}\int_{t_0}^{t_1}\big(l_1(\tau) - u(\tau,\xi)\big)' f_u'[\tau,x]R(x,\tau,s) \times \right.\right.$$
$$\times f_u[s,x]\big(l_1(s) - u(s,\xi)\big)ds d\tau +$$
$$+2\int_{t_0}^{t_1}\left[\int_t^{t_1}\big(l_1(\tau) - u(\tau,\xi)\big)' H_{uz}[\tau,\xi]V_{11}(\tau,\xi;t,\xi)d\tau\right] f_u[t,\xi] \times$$
$$\left.\left.\times\, \delta u(t,\xi)\big(l_1(t) - u(t,\xi)\big)dt + \int_{t_0}^{t_1}\big(l_1(t) - u(t,\xi)\big)' H_{uu}[t,\xi]\big(l(t) - u(t,\xi)\big)dt\right] + o(\varepsilon)\right\} +$$
$$o(\mu^2) \ge 0.$$

Hence we conclude that

$$E\left\{\int_{t_0}^{t_1}\int_{t_0}^{t_1}\big(l_1(\tau) - u(\tau,\xi)\big)' f_u'[\tau,x]R(x,\tau,s) \times \right.$$
$$\times f_u[s,x]\big(l_1(s) - u(s,\xi)\big)ds d\tau +$$
$$+2\int_{t_0}^{t_1}\left[\int_t^{t_1}\big(l_1(\tau) - u(\tau,\xi)\big)' H_{uz}[\tau,\xi]V_{11}(\tau,\xi;t,\xi)d\tau\right] f_u[t,\xi] \times$$
$$\left.\times\, \delta u(t,\xi)\big(l_1(t) - u(t,\xi)\big)dt + \int_{t_0}^{t_1}\big(l_1(t) - u(t,\xi)\big)' H_{uu}[t,\xi]\big(l(t) - u(t,\xi)\big)dt\right\} \le 0, \tag{21}$$

for all $\xi \in [x_0,x_1)$, $l_1(t) \in KC_r([t_0,t_1],R^r)$.

And if we determine the variation of the control $u(t,x)$ by the formula

$$\delta u_\varepsilon(t,x) = \begin{cases} l_2(x) - u(t,x), (t,x) \in D_\varepsilon = [\theta,\theta+\varepsilon) \times [x_0,x_1], \\ 0, (t,x) \in D/D_\varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a sufficiently small number, $\theta \in [t_0,t)$, $l_2(x) \in KC_r([x_0,x_1],R^r)$, then by symmetric reasoning it is proved that along the optimal process $\big(u(t,x), z(t,x), y(t,x)\big)$ the following equality takes place:

$$E\left\{\int_{x_0}^{x_1}\int_{x_0}^{x_1}(l(\tau)-u(\theta,\tau))'\,g_u[\theta,\tau]Q(\theta,\tau,s)g_u[s,\xi](l_1(s)-u(\theta,s))dsd\tau+\right.$$

$$+2\int_{x_0}^{x_1}\left[\int_{x}^{x_1}(l_2(\tau)-u(\theta,\tau))'\,H_{uy}[\theta,\tau]V_{22}(\theta,\tau;\theta,x)d\tau\right]\times$$

$$\times\, g_u[\theta,x]\big(l(x)-u(\theta,x)\big)dx+$$

$$\left.+\int_{x_0}^{x_1}\big(l_2(x)-u(\theta,x)\big)'H_{uu}[\theta,\xi]\big(l_2(x)-u(\theta,\xi)\big)dx\right\}\le 0, \qquad (22)$$

for all $\theta\in[t_0,t_1), l_2(x)\in KC_r([x_0,x],R^r)$.

The following is the final formulation of the results obtained for the optimality of quasi-singular controls.

Theorem 3. The optimality of the quasi-singular control $u(t,x)$ in considered stochastic control problem (1)-(3) requires that inequalities (21), (22) hold for all $\xi\in[x_0,x_1), l_1(t)\in KC_r([t_0,t_1],R^r)$, and $\theta\in[t_0,t_1), l_2(x)\in KC_r([x_0,x_1],R^r)$, respectively.

5. Conclusion

We study a stochastic optimal control problem described by systems of first-order nonlinear stochastic hyperbolic equations in canonical form. Under the assumption of convexity of control domains, necessary first- and second-order optimality conditions are established, and the optimality of quasi-singular controls is investigated.

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