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On stochastic process with fuzzy gamma distributed interference of chance and fuzzy exponential demands

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A B S T R A C T

In this study, stochastic process which describe a so called fuzzy-probabilistic inventory control model is considered. The fuzzy ergodic distribution of this process is obtained, when the amount of demand has exponential distribution and interference of chance has a gamma distribution with fuzzy parameters and the explicit formula for the n^{th}order moments of the fuzzy ergodic distribution was obtained.

1. Introduction

Modern probability theory is an ideal tool for modeling the stochastic nature of the real world. However, many problems that involve randomness cannot be modeled by the classical methods of probability theory due to the uncertainty of the concepts that arise when formulating the problem. Thus, for the first time, the concept of fuzzy probability measures were introduced by Lotfi Zadeh in [1, p.422]. Later, the concept of fuzzy random variables (FRVs) has evolved. The first definition was introduced by Kwakernaak [2, p.6], who viewed an FRV as a vague perception of a crisp but unobservable random variable (RV). Puri and Ralescu [3, p.413] proposed a different approach, defining FRVs as random fuzzy sets. Finally, Liu and Liu [4, p.146] introduced an alternative perspective, formulating FRVs based on a credibility measure. Same year and later in [5, p.501; 6, p.193; 7, p.200] Buckley and Eslami introduced both discrete and continuous case fuzzy probabilistic models, which effectively combined fuzzy set theory with traditional probability theory. This combination enabled the modeling of uncertainties that are not purely stochastic but also imprecise as commonly observed in real-world. In their discrete case model, Buckley and Eslami extended the concept of probability distributions to fuzzy environments, where probabilities were treated as fuzzy numbers. Similarly, their continuous case model replaced crisp probability density functions with fuzzy density functions. The practical applications of FRVs have been explored extensively, with Arnold F. Shapiro making significant contributions in this field. In [8, pp.307-314] Shapiro examined fuzzy random variables in the context of insurance and mathematical economics. Building on this, in [9, p.2686] Alireza Faraza and Arnold F. Shapiro applied FRVs to control charts, demonstrating their usefulness in quality control. In [10, pp.1-12] Shapiro provided preliminary observations on

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implementing FRVs, shedding light on practical challenges and potential solutions. He further advanced the field in [11, p.865] by modeling future lifetimes as fuzzy random variables, offering insights into actuarial applications. In [12, pp.248-253] Moussa, Kamdem and Terraza extended the application of FRVs to financial risk management, particularly in evaluating value-at-risk and expected shortfall for portfolios with heavy-tailed returns. Some researchers applied fuzzy probabilistic models to inventory control models. The foundational work by Aliyev, Khaniyev, and Aktaş [13, pp.273-276] introduced a fuzzy inventory model of type (s, S), where uncertainty in inventory parameters is modeled through fuzzy logic. Their approach assuming that the stock level ζ_n is almost surely constant [14, p.959; 15, p.532]. In our research, we propose a new extension of the fuzzy inventory model where the stock level is treated as a random variable with a fuzzy gamma distribution. This difference from the assumption of a constant stock level introduces a more realistic and adaptable framework for modeling inventory systems under uncertainty.

Recent works in the field by Alamri et.al [16, pp. 1-38] highlights the significance of considering fuzzy demand in fuzzy inventory control models. Other works like the intuitionistic fuzzy inventory model with a quadratic demand rate, time-dependent holding costs, and shortages, as proposed by Chaudhary and Kumar [17, pp. 1-12] provides strong framework for addressing complex supply chain dynamics. Additionally, models like "Optimizing Inventory Management: A Comprehensive Analysis of Models Integrating Diverse Fuzzy Demand Functions" by Mittal and Jail et.al [18] further emphasize the importance of diverse fuzzy demand functions in inventory management. These studies shows the growing importance of fuzzy inventory control models for improving supply chain management and efficiency in uncertain environment.

2. Primarily discussions and construction of process

Let, $\{\xi_n\}, \{\eta_n\}, \{\theta_n\}$ and $\{\zeta_n\}, n \geq 1$ are sequences of random variables defined on same probability space $(\Omega, \mathcal{F}, P)$, such that variables in each sequence independent and identically distributed. Suppose that ξ_n, η_n, θ_n and ζ_n can take only positive values and these distribution functions be denoted by [19, pp.74-77]

$$\begin{gathered} \Phi(t) = P\{\xi_1 \leq t\}, t > 0, F(x, \mu) = P\{\eta_1 \leq x\}, x > 0, \\ H(u) = P\{\theta_1 \leq u\}, u > 0, G(z, \lambda, \beta) = P\{\zeta_1 \leq z\}, z > 0. \end{gathered} \tag{1}$$

In this model, these random variables describes the stochastic inventory control model as follows:

η_n - represents the demand quantity, indicating the amount requested at each demand occurrence.

ξ_n- denotes the inter-arrival times between sequential demands.

θ_n- represents time that the process stays at level 0, representing the duration the system spends in an empty or inactive state.

ζ_n- the size of the jumps of the process $X(t)$ after it hits the level 0.

Now let us define independent renewal sequence $\{T_n\}$ and $\{Y_n\}$, $n \geq 1$ as follows using the initial sequences of the random variables $\{\xi_n\}$ and $\{\eta_n\}$, $n \geq 1$ as:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad Y_n = \sum_{i=1}^{n} \eta_i, \quad n = 1,2, \dots ; T_0 = Y_0 = 0. \tag{2}$$

Here random variables T_n and Y_n represents respectively the arrival time of the n^{th} demand and the total demand up to the n^{th} demand.

Let us also define sequence of integer valued random variables:

$$\begin{gathered} N_0 = 0\ ;\ N_1 = N(z) = \min\{k \geq 1 : z - Y_k < 0\}, \\ N_n \equiv N_n(\zeta_{n-1}) = min\{k \geq N_{n-1} + 1 : \zeta_{n-1} - \left(Y_k - Y_{N_{n-1}}\right) < 0\}, n = 2,3, \dots, \end{gathered} \tag{3}$$

Random variable N_n shows the number of demands before n^{th} replenishment of stock.

Let the random variables τ_n represents the n^{th} time of the process drops below the level 0 and γ_n represents the n^{th} moment of exit from the level 0:

$$\begin{gathered} \tau_0 = 0, \tau_1 = T_{N_1}, \gamma_0 = 0, \gamma_1 = \tau_1 + \theta_1, \\ \dots \\ \tau_n = \gamma_{n-1} + T_{N_n} - T_{N_{n-1}}, \quad \gamma_n = \tau_n + \theta_n, n = 1,2, \dots \end{gathered} \tag{4}$$

Define also the counting process $\nu(t)$ which describes the number of jumps of the process $X(t)$ in the interval $[0, t]$:

$$\nu(t) = max\{k \geq 0: T_k \leq t\}. \tag{5}$$

Thus the following stochastic process can be constructed using these notations:

$$X(t) = \max\{0, \zeta_n - Y_{\nu(t)} + Y_{N_n}\}, \gamma_n \leq t < \gamma_{n+1}, n = 0,1,2, \dots \tag{6}$$

Here $X(0) \equiv \zeta_0 = z$.
The stochastic process $X(t)$ describes the so-called probabilistic inventory control model. One of the trajectories of the process $X(t)$ is given in the following picture:

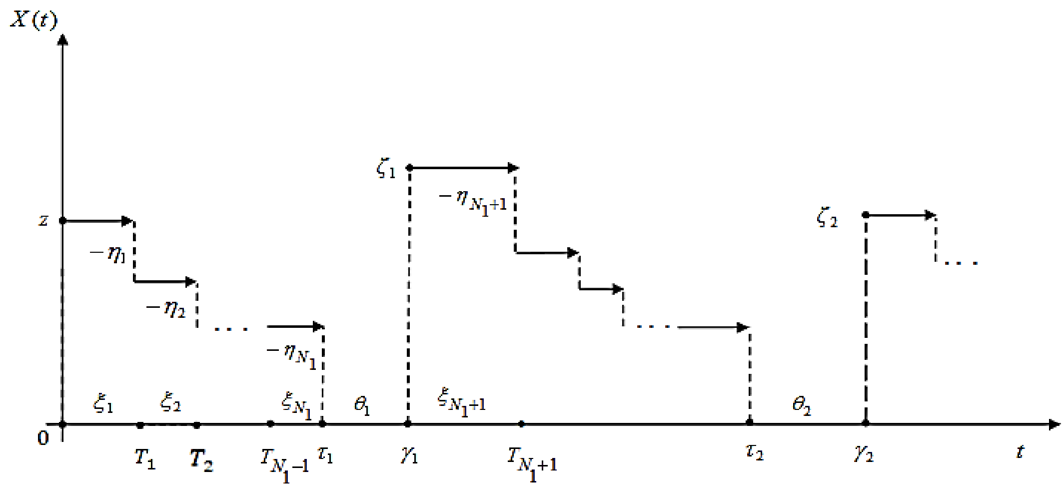

Fig. 1. One of the trajectories of the process $X(t)$

Proposition 2.1. Let initial sequence $\{\xi_n\}, \{\eta_n\}, \{\theta_n\}$ and $\{\zeta_n\}$, $n \geq 1$ – satisfies the following supplementary conditions:

1)$E\xi_1 < \infty$;

2) $E\theta_1 < \infty$;

3) Random variable η_1 has the exponential distribution with parameter $\mu > 0$;

4) Random variable $\zeta_1, n \geq 1$ has the gamma distribution with parameters (β, λ).

Then the process $X(t)$ is ergodic and ergodic distribution function has the following explicit form:

$$Q_X(x, \mu, \beta, \lambda) = 1 - \frac{\mu x}{\lambda + \beta\mu + \lambda K} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu - \lambda\mu x}{\lambda + \beta\mu + \lambda K}\left(1 - G_{\beta,\lambda}(x)\right),$$

where

$$g_{\beta,\lambda}(x) = \frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x}, \; G_{\beta,\lambda}(x) = \frac{\lambda^\beta}{\Gamma(\beta)} \int_0^x t^{\beta-1} e^{-\lambda t} \, dt,$$

and $K = \frac{E\theta_1}{E\xi_1}$ is delay coefficent.

Proof. Considered process belongs to a wide class of the processes which is called as "Processes with a discrete interference of chance" in literature. For this class, the general ergodic theorem is given in monograph Gihman and Skorohod [20, p.244]. Conditions 1)-4) of this theorem provide the fulfillment of the conditions of the general ergodic theorem.

3. Main results

In this section we will assume that random variable η_1 has fuzzy exponential distribution with parameter $\tilde{\mu} > 0$ and random variable ζ_1 has fuzzy gamma distribution with fuzzy parameters $(\tilde{\beta}, \tilde{\lambda})$ and under this assumption expicit formula for $n^{th}, n \geq 1$ order moments of fuzzy ergodic distribution found. We will use the concepts of fuzzy sets, fuzzy numbers and fuzzy distributions. Which can be easily found well known literature [21, pp.7-28; 22, pp.5-15; 23, pp.1619-1613; 24, pp. 97-117; 25, pp.11-43].

Theorem 3.1. Let the conditions of Proposition 2.1 be satisfied. Additionally let us assume that $K = 0$ and that $\tilde{\mu} > 0$ is an arbitrary fuzzy number.Then

$$\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda)[\alpha] = [Q_1^\alpha(x), Q_2^\alpha(x)], \alpha \in [0,1],$$

where

$$Q_1^\alpha(x) = 1 - \frac{\mu^\alpha x}{\lambda + \beta\mu^\alpha} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu^\alpha - \lambda\mu^\alpha x}{\lambda + \beta\mu^\alpha}\Big(1 - G_{\beta,\lambda}(x)\Big),$$

$$Q_2^\alpha(x) = 1 - \frac{\mu_\alpha x}{\lambda + \beta\mu_\alpha} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu_\alpha - \lambda\mu_\alpha x}{\lambda + \beta\mu_\alpha}\Big(1 - G_{\beta,\lambda}(x)\Big).$$

Proof. It's obvious that for any $x > 0$,

$$\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda)[\alpha] = \{Q_X(x, \mu, \beta, \lambda) | \mu \in \tilde{\mu}[\alpha]\} = \left[\min_{\mu\in\tilde{\mu}[\alpha]} Q_X(x, \mu, \beta, \lambda), \max_{\mu\in\tilde{\mu}[\alpha]} Q_X(x, \mu, \beta, \lambda)\right].$$

Where

$$Q_X(x, \mu, \beta, \lambda) = 1 - \frac{\mu x}{\lambda + \beta\mu} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu - \lambda\mu x}{\lambda + \beta\mu}\Big(1 - G_{\beta,\lambda}(x)\Big).$$

Lets denote $Q_1^\alpha(x) = \min_{\mu\in\tilde{\mu}[\alpha]} Q_X(x, \mu, \beta, \lambda)$ and $Q_2^\alpha(x) = \max_{\mu\in\tilde{\mu}[\alpha]} Q_X(x, \mu, \beta, \lambda)$. Our aim is to calculate $Q_1^\alpha(x)$ and $Q_2^\alpha(x)$ for any $x > 0$. For this purpose, we need to investigate the monotonicity of the function $Q_X(x, \mu, \beta, \lambda) = Q(\mu)$ on the interval $\tilde{\mu}[\alpha]$.

$$\frac{dQ(\mu)}{d\mu} = -\frac{\lambda x}{(\lambda + \beta\mu)^2} g_{\beta,\lambda}(x) - \frac{\lambda^2 x}{(\lambda + \beta\mu)^2}\Big(1 - G_{\beta,\lambda}(x)\Big) =$$

$$= -\frac{\lambda x\left[g_{\beta,\lambda}(x) + \lambda\Big(1 - G_{\beta,\lambda}(x)\Big)\right]}{(\lambda + \beta\mu)^2} < 0.$$

Therefore, since the function $Q(\mu)$ decreases on the interval $\tilde{\mu}[\alpha] = [\mu_\alpha, \mu^\alpha]$, it will take its smallest and largest values at the end points μ^α and μ_α of this interval, respectively. Then

$$Q_1^\alpha(x) = Q_X(x, \mu^\alpha, \beta, \lambda) = 1 - \frac{\mu^\alpha x}{\lambda + \beta\mu^\alpha} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu^\alpha - \lambda\mu^\alpha x}{\lambda + \beta\mu^\alpha}\Big(1 - G_{\beta,\lambda}(x)\Big),$$

$$Q_2^\alpha(x) = Q_X(x, \mu_\alpha, \beta, \lambda) = 1 - \frac{\mu_\alpha x}{\lambda + \beta\mu_\alpha} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu_\alpha - \lambda\mu_\alpha x}{\lambda + \beta\mu_\alpha}\Big(1 - G_{\beta,\lambda}(x)\Big).$$

Theorem 3.1 is proved.

Theorem 3.2. Let conditions of Theorem 3.1 be satisfied. Assume that $\tilde{\mu} = (\mu_1/\mu_2/\mu_3) > 0$ is a triangular fuzzy number. Then the fuzzy ergodic distribution $\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda)$ has the following approximate triangular form:

$$\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda) \cong (Q_1(x)/Q_2(x)/Q_3(x)).$$

where

$$Q_1(x) = Q_1^0(x) = Q_X(x, \mu_3, \beta, \lambda) = 1 - \frac{\mu_3 x}{\lambda + \beta\mu_3} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu_3 - \lambda\mu_3 x}{\lambda + \beta\mu_3}\left(1 - G_{\beta,\lambda}(x)\right),$$

$$Q_2(x) = Q_X(x, \mu_2, \beta, \lambda) = 1 - \frac{\mu_2 x}{\lambda + \beta\mu_2} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu_2 - \lambda\mu_2 x}{\lambda + \beta\mu_2}\left(1 - G_{\beta,\lambda}(x)\right),$$

$$Q_3(x) = Q_2^0(x) = Q_X(x, \mu_1, \beta, \lambda) = 1 - \frac{\mu_1 x}{\lambda + \beta\mu_1} g_{\beta,\lambda}(x) + \frac{\lambda + \beta\mu_1 - \lambda\mu_1 x}{\lambda + \beta\mu_1}\left(1 - G_{\beta,\lambda}(x)\right).$$

Proof. According to Theorem 3.1

$$\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda)[0] = [Q_1^0(x), Q_2^0(x)],$$

where

$$Q_1^0(x) = \min_{\mu \in [\mu_1, \mu_3]} Q_X(x, \mu, \beta, \lambda) = Q_X(x, \mu_3, \beta, \lambda),$$

$$Q_2^0(x) = \max_{\mu \in [\mu_1, \mu_3]} Q_X(x, \mu, \beta, \lambda) = Q_X(x, \mu_1, \beta, \lambda).$$

and because

$$\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda)[1] = [Q_1^1(x), Q_2^1(x)] = [Q_X(x, \mu_2, \beta, \lambda), Q_X(x, \mu_2, \beta, \lambda)] = Q_X(x, \mu_2, \beta, \lambda),$$

we obtain the following natural approximate triangular form for the fuzzy distribution $\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda)$:

$$\tilde{Q}_X(x, \tilde{\mu}, \beta, \lambda) \cong (Q_1(x)/Q_2(x)/Q_3(x)).$$

Here

$$Q_1(x) = Q_1^0(x) = Q_X(x, \mu_3, \beta, \lambda),$$

$$Q_2(x) = Q_X(x, \mu_2, \beta, \lambda),$$

$$Q_3(x) = Q_2^0(x) = Q_X(x, \mu_1, \beta, \lambda).$$

Theorem 3.2 is proved.

Example 3.1. Lets suppose that $\lambda = \beta = 1$ and $\tilde{\mu} = (1/2/3)$. Then its obvious that

$$\mu[\alpha] = [1 + \alpha, 3 - \alpha], \alpha \in [0,1],$$

and

$$g_{1,1}(x) = e^{-x}, G_{1,1}(x) = 1 - e^{-x}.$$

Under these conditions, let us find the fuzzy number $\tilde{Q}_X(1, \tilde{\mu}, 1,1)$ and its approximate triangular form when $x = 1$. It is clear that,

$$\tilde{Q}_X(1, \tilde{\mu}, 1,1)[\alpha] = [Q_1^\alpha(1), Q_2^\alpha(1)],$$

where

$$Q_1^\alpha(1) = min\left\{\frac{1 + e^{-1} + \mu(1 - e^{-1})}{1 + \mu}\middle|\mu \in [1 + \alpha, 3 - \alpha]\right\} = \frac{1 + e^{-1} + (3 - \alpha)(1 - e^{-1})}{4 - \alpha},$$

$$Q_2^{\alpha}(1) = max\left\{\frac{1+e^{-1}+\mu(1-e^{-1})}{1+\mu}\middle|\mu \in [1+\alpha, 3-\alpha]\right\} = \frac{1+e^{-1}+(1+\alpha)(1-e^{-1})}{2+\alpha}.$$

On the other hand

$$Q_1(1) = Q_1^0(1) \approx\ 0.816,$$
$$Q_2(1) = Q_1^1(1) = Q_2^1(1) \approx 0.877,$$
$$Q_3(1) = Q_2^0(1) =\ 1.$$

Then

$$\tilde{Q}_X(1, \tilde{\mu}, 1,1) \cong (Q_1(1)/Q_2(1)/Q_3(1)) = (0.816/0.877/1).$$

The difference between the fuzzy numbers $\tilde{Q}_X(1, \tilde{\mu}, 1,1)$ and $(0.816/0.877/1)$ can be seen from the following graph:

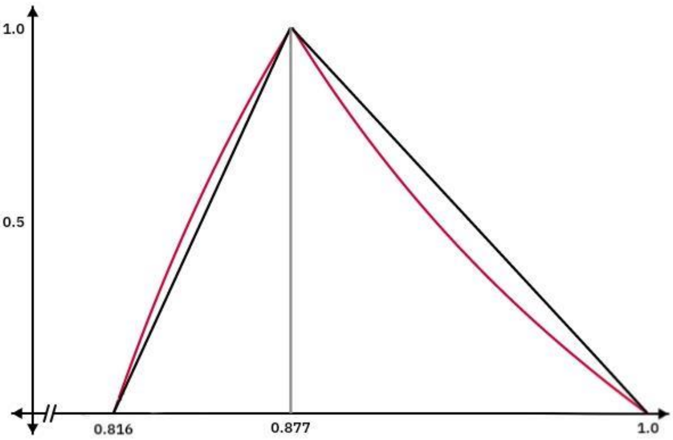

Fig. 2. Membership functions of fuzzy number $\tilde{Q}_X(1, \tilde{\mu}, 1,1)$ and $(0.816/0.877/1)$

Let denote $n^{th}, n \geq 1$ order moments of fuzzy ergodic distribution $\tilde{Q}_X(x, \tilde{\mu}, \tilde{\beta}, \tilde{\lambda})$ as $\tilde{E}X^n$, which is the fuzzy extension of

$$EX^n = \int_0^{\infty} x^n\, q(x)dx,$$

where $q(x) = q(x, \mu, \beta, \lambda)$ ergodic density function of the process $X(t)$.

Theorem 3.3. Let the conditions of Proposition 2.1 be satisfied. Additionally let us assume that random variable η_1 has fuzzy exponential distribution with parameter $\tilde{\mu} > 0$, random variables ζ_1 has fuzzy gamma distribution with parameters $(\tilde{\beta}, \tilde{\lambda})$ and fuzzy ergodic density function of the process $X(t)$ exist. Then $\tilde{E}X^n$ has the following explicit form:

$$\tilde{E}X^n = -\frac{\tilde{\mu}}{\tilde{\lambda}+\tilde{\beta}\tilde{\mu}+\tilde{\lambda}\tilde{K}}\left(\frac{\Gamma(\tilde{\beta}+n)}{\Gamma(\tilde{\beta})\tilde{\lambda}^n}+\frac{\Gamma(\tilde{\beta}+n)}{\Gamma(\tilde{\beta}-1)\tilde{\lambda}^n}-\frac{2\Gamma(\tilde{\beta}+n+1)}{\Gamma(\tilde{\beta})\tilde{\lambda}^n}+\frac{\Gamma(\tilde{\beta}+2)}{2\Gamma(\tilde{\beta})\tilde{\lambda}}\right).$$

Proof. We will prove the theorem using α-cuts method. First let's find fuzzy ergodic density function $\tilde{q}(x, \tilde{\mu}, \tilde{\beta}, \tilde{\lambda})$ of process $X(t)$ as:

$$\tilde{q}(x, \tilde{\mu}, \tilde{\beta}, \tilde{\lambda})\ = \frac{d}{dx}\tilde{Q}_X(x, \tilde{\mu}, \tilde{\beta}, \tilde{\lambda}).$$

Then,

$$\tilde{q}(x, \tilde{\mu}, \tilde{\beta}, \tilde{\lambda})[\alpha]\ = \left(\frac{d}{dx}\tilde{Q}_X(x, \tilde{\mu}, \tilde{\beta}, \tilde{\lambda})\right)[\alpha] = \left\{\frac{d}{dx}Q_X(x, \mu, \beta, \lambda)\middle|\mu \in \mu[\alpha], \beta \in \beta[\alpha], \lambda \in \lambda[\alpha]\right\} =$$

$$= \left\{\frac{1}{\lambda+\beta\mu+\lambda K}\left(-\mu g_{\beta,\lambda}(x)-\mu x\frac{d}{dx}g_{\beta,\lambda}(x)-\lambda\mu\left(1-G_{\beta,\lambda}(x)\right)+\lambda\mu x g_{\beta,\lambda}(x)\right)\middle|\begin{matrix}\mu\in\mu[\alpha],\beta\in\beta[\alpha],\\ \lambda\in\lambda[\alpha]\end{matrix}\right\} =$$

$$= \left\{-\frac{\mu}{\lambda+\beta\mu+\lambda K}\left(g_{\beta,\lambda}(x)+x\frac{d}{dx}g_{\beta,\lambda}(x)+\lambda\left(1-G_{\beta,\lambda}(x)\right)-\lambda x g_{\beta,\lambda}(x)\right)\middle|\begin{matrix}\mu\in\mu[\alpha],\beta\in\beta[\alpha],\\ \lambda\in\lambda[\alpha]\end{matrix}\right\}.$$

On the other hand, because

$$\frac{d}{dx}g_{\beta,\lambda}(x)=\lambda g_{\beta-1,\lambda}(x)-\lambda g_{\beta,\lambda}(x),$$

then $\tilde{q}\left(x,\tilde{\mu},\tilde{\beta},\tilde{\lambda}\right)[\alpha]$ has the following form,

$$\tilde{q}\left(x,\tilde{\mu},\tilde{\beta},\tilde{\lambda}\right)[\alpha] =$$

$$= \left\{-\frac{\mu}{\lambda+\beta\mu+\lambda K}\left(g_{\beta,\lambda}(x)+\lambda x g_{\beta-1,\lambda}(x)-\lambda x g_{\beta,\lambda}(x)+\lambda\left(1-G_{\beta,\lambda}(x)\right)-\lambda x g_{\beta,\lambda}(x)\right)\middle|\begin{matrix}\mu\in\mu[\alpha],\\ \beta\in\beta[\alpha],\\ \lambda\in\lambda[\alpha].\end{matrix}\right\}$$

$$= \left\{-\frac{\mu}{\lambda+\beta\mu+\lambda K}\left(g_{\beta,\lambda}(x)+\lambda x g_{\beta-1,\lambda}(x)-2\lambda x g_{\beta,\lambda}(x)+\lambda\left(1-G_{\beta,\lambda}(x)\right)\right)\middle|\begin{matrix}\mu\in\mu[\alpha],\\ \beta\in\beta[\alpha],\\ \lambda\in\lambda[\alpha].\end{matrix}\right\}.$$

Using this α-cut of $\tilde{q}(x)=\tilde{q}\left(x,\tilde{\mu},\tilde{\beta},\tilde{\lambda}\right)$ we can observe that fuzzy ergodic density function has following explicit form,

$$\tilde{q}(x)=-\frac{\tilde{\mu}}{\tilde{\lambda}+\tilde{\beta}\tilde{\mu}+\tilde{\lambda}K}\left(g_{\tilde{\beta},\tilde{\lambda}}(x)+\tilde{\lambda}x g_{\beta-1,\lambda}(x)-2\tilde{\lambda}x g_{\tilde{\beta},\tilde{\lambda}}(x)+\tilde{\lambda}\left(1-G_{\tilde{\beta},\tilde{\lambda}}(x)\right)\right).$$

Now we can calculate using α-cuts of $\tilde{E}X^n=\int_0^\infty x^n\,\tilde{q}(x)dx$, n^{th} order moment of ergodic distribution:

$$(EX^n)[\alpha]=\left\{-\frac{\mu}{\lambda+\beta\mu+\lambda K}\left(\begin{matrix}\int_0^\infty x^n g_{\beta,\lambda}(x)dx+\lambda\int_0^\infty x^{n+1}g_{\beta-1,\lambda}(x)dx-\\ -2\lambda\int_0^\infty x^{n+1}g_{\beta,\lambda}(x)dx+\lambda\int_0^\infty x\left(1-G_{\beta,\lambda}(x)\right)dx\end{matrix}\right)\middle|\begin{matrix}\mu\in\mu[\alpha],\\ \beta\in\beta[\alpha],\\ \lambda\in\lambda[\alpha].\end{matrix}\right\}.$$

n^{th} order moment of gamma distribution has following form:

$$\int_0^\infty x^n g_{\beta,\lambda}(x)dx=\frac{\Gamma(\beta+n)}{\Gamma(\beta)\lambda^n}.$$

and

$$1-G_{\beta,\lambda}(x)=1-\int_0^x\frac{\lambda^\beta}{\Gamma(\beta)}t^{\beta-1}e^{-\lambda t}\,dt=\int_x^\infty\frac{\lambda^\beta}{\Gamma(\beta)}t^{\beta-1}e^{-\lambda t}\,dt=\frac{\Gamma(\beta,\lambda x)}{\Gamma(\beta)},$$

where $\Gamma(\beta,\lambda x)$ incomplete gamma function. Then

$$(EX^n)[\alpha]=$$

$$=\left\{-\frac{\mu}{\lambda+\beta\mu+\lambda K}\left(\frac{\Gamma(\beta+n)}{\Gamma(\beta)\lambda^n}+\lambda\frac{\Gamma(\beta+n)}{\Gamma(\beta-1)\lambda^{n+1}}-2\lambda\frac{\Gamma(\beta+n+1)}{\Gamma(\beta)\lambda^{n+1}}+\frac{\lambda}{\Gamma(\beta)}\int_0^\infty x\Gamma(\beta,\lambda x)dx\right)\middle|\begin{matrix}\mu\in\mu[\alpha],\\ \beta\in\beta[\alpha],\\ \lambda\in\lambda[\alpha].\end{matrix}\right\}.$$

It’s well known that for incomplete gamma functions,

$$\int_0^\infty z^{a-1}\Gamma(b,z)dz=\frac{\Gamma(a+b)}{a}.$$

Here if we take $z=\lambda x, a=2$, and $b=\beta$,

$$\int_0^\infty x\Gamma(\beta,\lambda x)dx = \frac{\Gamma(\beta+2)}{2\lambda^2}.$$

Finally, we have

$$(EX^n)[\alpha] =$$
$$= \left\{ -\frac{\mu}{\lambda+\beta\mu+\lambda K}\left[\frac{\Gamma(\beta+n)}{\Gamma(\beta)\lambda^n} + \frac{\Gamma(\beta+n)}{\Gamma(\beta-1)\lambda^n} - \frac{2\Gamma(\beta+n+1)}{\Gamma(\beta)\lambda^n} + \frac{\Gamma(\beta+2)}{2\Gamma(\beta)\lambda}\right] \middle| \begin{array}{l}\mu\in\mu[\alpha],\\ \beta\in\beta[\alpha],\\ \lambda\in\lambda[\alpha].\end{array} \right\}.$$

It shows that $\tilde{E}X^n$ has the following explicit form:

$$\tilde{E}X^n = -\frac{\tilde{\mu}}{\tilde{\lambda}+\tilde{\beta}\tilde{\mu}+\tilde{\lambda}K}\left(\frac{\Gamma(\tilde{\beta}+n)}{\Gamma(\tilde{\beta})\tilde{\lambda}^n} + \frac{\Gamma(\tilde{\beta}+n)}{\Gamma(\tilde{\beta}-1)\tilde{\lambda}^n} - \frac{2\Gamma(\tilde{\beta}+n+1)}{\Gamma(\tilde{\beta})\tilde{\lambda}^n} + \frac{\Gamma(\tilde{\beta}+2)}{2\Gamma(\tilde{\beta})\tilde{\lambda}}\right).$$

Theorem 3.3 is proved.

The n^{th} order moments describe the shape of the ergodic distribution of the process which is important for studying statistical properties of process. The first moment (mean) gives the central tendency, the second moment (variance) measures spread, and higher moments (skewness, kurtosis) capture asymmetry and tail behavior.

Example 3.2. Let's suppose that $K = 0, \lambda = 1, \beta = 2$ and $\tilde{\mu} = (1/2/3)$. Then its obvious that

$$\mu[\alpha] = [1+\alpha, 3-\alpha], \alpha \in [0,1].$$

Under these conditions, let us find the first moment $\tilde{E}X$ of ergodic distribution. It is clear that, according to Theorem 3.3

$$(\tilde{E}X)[\alpha] = \left[(\tilde{E}X)_\alpha, (\tilde{E}X)^\alpha\right], \alpha \in [0,1].$$

Where,

$$(\tilde{E}X)_\alpha = min\left\{\frac{5\mu}{1+2\mu}\middle|\mu\in[1+\alpha,3-\alpha]\right\} = \frac{5(1+\alpha)}{3+2\alpha},$$
$$(\tilde{E}X)^\alpha = max\left\{\frac{5\mu}{1+2\mu}\middle|\mu\in[1+\alpha,3-\alpha]\right\} = \frac{5(3-\alpha)}{7-2\,\alpha}.$$

Using this α-cuts we can easly find that first moment $\tilde{E}X$ of ergodic distribution is following fuzzy number with membership function

$$\mu_{\tilde{E}X}(x) = \begin{cases} 0, & x < \dfrac{5}{3} \\ \dfrac{5-3x}{2x-5}, & \dfrac{5}{3} \le x \le 2 \\ \dfrac{7x-15}{2x-5}, & 2 \le x \le \dfrac{15}{7}. \\ 0, & x \ge \dfrac{15}{7} \end{cases}$$

The graph of membership function of first moment is an asymmetric triangular membership function with non-linear edges,

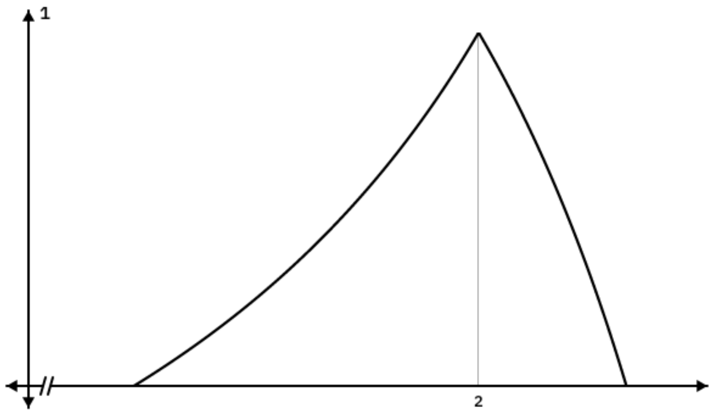

Fig. 3. Membership function of corresponding first moment

4. Conclusion

In conclusion, fuzzy inventory control models play an important role in managing uncertainty and variability in real-world systems. Important aspect of these models is the considering fuzzy demand, which reflects the imprecise and often unpredictable nature of customer needs. By acknowledging this uncertainty, decision-makers can optimize stock levels, reduce costs, and improve service quality. A potential work for future research is the exploration of stochastic processes with general fuzzy interference of chance and general fuzzy demands. Furthermore, analyzing how fuzzy heavy and light-tailed distributed interference of chance can affect the ergodic distribution of stochastic process. These improvements would help to create stronger and more flexible fuzzy inventory models, making them better at managing uncertainty and randomness in practice.

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