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On application of a semi-Markov random walk process in logistics

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ABSTRACT

In this work, we will investigate a semi-Markov random walk process for warehouse management in logistics. We consider generating function for a boundary functional for semi-Markov random walk process. By the generating function we can find moments the distribution of warehouse level over time in the context of a semi-Markov random walk process. The random variable is introduced. This random variable representing, the number of steps required to reach a positive level. Here the length of each step follows a gamma distribution. The generating function of the distribution of the random variable is expressed as an integral equation. The purpose of the paper is to reduce the fractional order integral equation to a fractional order differential equation. Finally, the paper aims to derive an explicit form of the generating function.

1. Introduction

Using semi-Markov random walk processes can be modeled the movement of goods within a warehouse or fulfillment center, including picking, packing, and shipping operations. Each step in the random walk represents a movement of goods from one location to another, with transition probabilities and durations determined by factors such as order processing times, storage layout, and equipment availability. By simulating warehouse operations as semi-Markov random walks, we can optimize resource allocation, and order picking strategies to minimize cycle times and improve order fulfillment efficiency.

The generating function for a boundary functional in the context of a random walk process is a powerful tool to analyze various properties of the semi-Markov process. The authors of studied (see, [1], [2], [4], [7]) have conducted research on the generating function of the distribution of the boundary functional of a semi-Markovian random walk. This topic is of interest in stochastic processes and probability theory, particularly in understanding the behavior of random walks and related processes. It should be noted that finding the generation function of semi-Markov random walk processes helps to learn the characteristics of a probability distribution of the process. By analyzing the properties of the generation function, such as its moments, logistics analysts can gain insights into the stochastic behavior of the system. This analysis helps in understanding the variability and uncertainty inherent in logistics processes, enabling better decision-making. Its applicability make, it usefully for understanding complex systems and making informed decisions in real process. It is well known the connection between semigroup theory and the Markov processes. In the

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semigroup theory of Markov processes, a particular process is usually represented as a semigroup of contraction operators in some concrete Banach space, and the properties of the particular process are deduced from the properties of the associated semigroup of operators. From this point of view, by Atangana in [5] it was shown that the Atangana Baleanu fractional derivative possesses the Markovian and non-Markovian properties. We also refer to [6] for more results on fractional modeling of probabilistic processes. We recall that, in [3] the authors studied a stepwise semi-Markovian processes. Then authors used the fractional Riemann-Liouville derivative. Moreover, the obtained solution for the fractional differential equation was in the form of a threefold sum. But in the presented paper, we obtained a mathematical model of a semi-Markovian process for a certain general class of probability distributions, and in the class of gamma distributions we managed to reduce the study of a mathematical model through a fractional differential equation with a fractional Weyl derivative. In conclusion, we were able to find an explicit form of the solution of the resulting fractional differential equation and hence, the exact probabilistic characteristics of the process under consideration. In this paper, jump processes with a waiting time between jumps that is not necessarily given by an exponential random variable is consider. In the present paper, we study the semi-Markov random walk processes with negative drift, positive jumps. We construct an integral equation for the generating function of the conditional distribution of the numbers of the steps for the first moment of reaching positive level. In particular, constructed integral equation is reduced to the fractional order differential equation in the class of gamma distributions.

2. Problem statement

Let there be given a sequence of independent and identically distributed pairs of random variables $\{\xi_k, \zeta_k\}_{k\geq 1}$ defined on the probability space $(\Omega, \mathrm{F}, \mathrm{P})$, where the random variables ξ_k and ζ_k, $k = \overline{1, \infty}$ are positive valued, i.e,

$$P\{\xi_k > 0\} = 1, P\{\zeta_k > 0\} = 1.$$

In addition, the random variables ξ_k and ζ_k, $k = \overline{1, \infty}$ are mutually independent as well. Let us denote the cumulative distribution functions of the random variables ξ_1 and ζ_1

$$F(t) = P\{\xi_1 < t\} \text{ and } \mathrm{G(t)} = P\{\zeta_k < t\}. \ \mathrm{t} \in R^+$$

respectively.

Using these random variables, we construct the following semi-Markov random walk process:

$$X_z(t) = z - t + \sum_{i=0}^{k-1} \zeta_i, \quad \text{if} \quad \sum_{i=0}^{k-1} \xi_i \leq t < \sum_{i=0}^{k} \xi_i \,,$$

$$k = \overline{1, \infty},$$

where $\xi_0 = \zeta_0 = 0$. $X_z(t)$ is called the semi-Markov random walk process with negative drift, positive jumps.

One of the realization of process $X_z(t)$ is shown in Figure 1.

Here ν_1^a is the random variable representing the numbers of the steps for the first time the random walk process reaches the boundary:

$$\nu_1^a = \min\left\{k: z - \sum_{i=1}^{k} [\xi_i - \zeta_i] \geq a\right\}$$

where we assume that $\zeta_i - \xi_i > 0$ with probability 1.

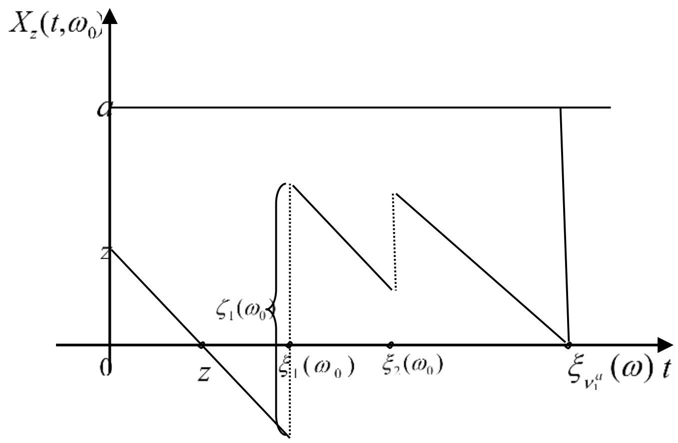

Fig. 1. The semi-Markovian random walk process $X_z(t)$

The aim of the present work is to determine the generating function of the conditional distribution of the random variable ν_1^a.

3. Generating function of the conditional distribution ν_1^a

The generating function of the conditional distribution of the boundary functional can be defined as follows:

$$\psi\,(u\mid z)=E\left(u^{\nu_1^a}\middle|X_z(0)=z\right)=\sum_{k=1}^{\infty}u^kP\{\nu_1^a=k|X_z(0)=z\},$$
$$0<u\le 1.$$

If we use the law of total probability in the context of the generating function $\psi\,(u\mid z)$ and the random variable ν_1^a representing the number of steps for the first moment of reaching a positive level, we get:

$$\begin{aligned}\psi(u|z)=\ &u\int_0^\infty p_{\xi_1}(x)P\{\zeta_1>a-z+x\}dx+\\&+u\int_z^a\psi(u|y)\int_0^\infty d_yP\{\zeta_1<x+y-z\}\,d_xP\{\xi_1<x\}+\\&+u\int_{-\infty}^z\psi(u|y)\int_{z-y}^\infty d_yP\{\zeta_1<x+y-z\}\,d_xP\{\xi_1<x\}.\end{aligned}\tag{1}$$

If $p_{\xi_1}(t)$ and $p_{\zeta_1}(t)$ are the probability density functions, then the integral equation (1) for the generating function $\psi(u|z)$ will be take the form:

$$\begin{aligned}\psi(u|z)=\ &u\int_0^\infty p_{\xi_1}(x)P\{\zeta_1>a-z+x\}dx+\\&+u\int_z^a\psi(u|y)\int_0^\infty p_{\zeta_1}(x+y-z)p_{\xi_1}(x)dxdy+\\&+u\int_{-\infty}^z\psi(u|y)\int_{z-y}^\infty p_{\zeta_1}(x+y-z)p_{\xi_1}(x)dxdy.\end{aligned}\tag{2}$$

In the case of a gamma distribution with parameters $\alpha>0$ and $\beta>0$, $p_{\xi_1}(t)$ is given by:

$$\rho_{\xi_1}(t)=\begin{cases}\dfrac{\beta^\alpha}{\Gamma(\alpha)}\cdot t^{\alpha-1}\,e^{-\beta t}, & t>0,\\ 0, & t\le 0,\end{cases}$$

where $\Gamma(\alpha)$ is the gamma function.

In the case of a exponential distribution with parameter λ, $p_{\zeta_1}(t)$ is given by:

$$\rho_{\zeta_1}(t) = \begin{cases} \lambda\, e^{-\lambda t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Therefore, integral equation (2) for the generating function $\psi(u|z)$ has the form:

$$\begin{aligned} \psi(u\,|z) = {} & \frac{u\beta^{\alpha}e^{-\lambda(a-z)}}{(\lambda+\beta)^{\alpha}} + \\ & + \frac{u\lambda\beta^{\alpha}e^{\lambda z}}{\Gamma(\alpha)} \int_{z}^{a} e^{-\lambda y}\psi(u|y) \int_{0}^{\infty} e^{-(\lambda+\beta)x}x^{\alpha-1}dxdy + \\ & + \frac{u\lambda\beta^{\alpha}e^{\lambda z}}{\Gamma(\alpha)} \int_{-\infty}^{z} e^{-\lambda y}\psi(u|y) \int_{z-y}^{\infty} e^{-(\lambda+\beta)x}x^{\alpha-1}dxdy. \end{aligned} \tag{3}$$

After some transformations, the (3) integral equation takes the form

$$\begin{aligned} & e^{\beta z}\psi'(u\,|z) - \lambda\, e^{\beta z}\psi(u\,|z) = \\ & = -\frac{u\lambda\beta^{\alpha}}{\Gamma(\alpha)} \int_{-\infty}^{z} e^{\beta y}\psi(u|y)(z-y)^{\alpha-1}\, dy. \end{aligned} \tag{4}$$

We set

$$Q(u|z) = e^{\beta z}\psi(u\,|z). \tag{5}$$

From equation (4), we obtain

$$\begin{aligned} & Q'_z(u\,|z) - (\lambda+\beta)\, Q(u\,|z) = \\ & = -\frac{u\beta^{\alpha}\lambda}{\Gamma(\alpha)} \int_{-\infty}^{z} Q(u\,|y)(z-y)^{\alpha-1}\, du. \end{aligned}$$

It is known that The general form of the Weyl fractional integral is given by:

$$I^{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-y)^{\alpha-1} f(y)\, dy.$$

where α is a real number , Γ is the gamma function, and f is the function being integrated.

Taking into account the last equality, we have

$$Q'_z(u|z) - (\lambda+\beta)\, Q(u|z) + u\beta^{\alpha}\lambda I^{\alpha}(Q(u|z)) = 0. \tag{6}$$

Let $\alpha \in (0,1)$ and if the function $f(x)$ is differentiable in $(0,\infty)$. Then the Weyl fractional derivative of order α as:

$$D^{\alpha}(f(x)) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} (x-y)^{-\alpha} f'(y) dy.$$

Applying to both sides equation (6) the Weyl fractional derivative of order α we get

$$\begin{aligned} & D_z^{\alpha+1}\big(Q(u\,|z)\big) - (\lambda+\beta) D_z^{\alpha}\big(Q(u\,|z)\big) + \\ & + \lambda\, \beta^{\alpha} u Q(u\,|z) = 0. \end{aligned} \tag{7}$$

4. Solution of equation (7)

Suppose that $k(u)$ is a real function defined on $(0,\infty)$. Then we can find the Weyl fractional derivative of order α of the function $e^{k(u)\,z}$ with respect to z:

$$D_z^{\alpha}\, e^{k(u)\,z} = [k(u)]^{\alpha} e^{k(u)\,z}.$$

Let us seek the solution of fractional order differential equation (7) in the form $Q(u\,|z) = C(u)e^{k(u)\,z}$. Here $k(u)$ is a solution of the characteristic equation of (7) with fractional exponents

$$[k(u)]^{\alpha+1} - (\mu + \beta) \cdot [k(u)]^{\alpha} - \mu\, \beta^{\alpha} u = 0.$$

and the unknown function $C(u)$can be determined from initial condition

$$\psi(u\,|0) = \frac{u\beta^{\alpha} e^{-\lambda a}}{(\lambda+\beta)^{\alpha}} + \frac{u\lambda\beta^{\alpha}}{(\lambda+\beta)^{\alpha}} \int_0^a e^{-\lambda\, y} \psi(u|y) dy +$$

$$+ \frac{u\lambda\beta^{\alpha}}{\Gamma(\alpha)} \int_{-\infty}^{0} e^{-\lambda\, y} \psi(u|y) \int_{-y}^{\infty} e^{-(\lambda+\beta)x} x^{\alpha-1} dx dy\,.$$

From (5) we have

$$\psi(u|z) = C(u) e^{(k(u)-\beta)\, z}.$$

It is obvious that $\psi(1|z) = 1$. Hence, we obtain that $C(1) = 1$ and $k(1) = \beta$.

5. Conclusion

The main goal of the paper is to investigate semi-Markov random walk process semi-Markov random walk process with negative drift and positive jumps for modeling the warehouse management in logistics. An integral equation for the generating function of the conditional distribution of the numbers of steps required for the first moment of reaching a positive level is constructed. In particular, the integral equation derived for the generating function is reduced to a fractional order differential equation within the class of gamma distributions. The paper employs fractional derivative in the Weyl sense to describe the fractional order differential equation. Finally, we find an exact form for generating function for investigate characteristics of a probability distribution of warehouse level.

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