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Qualitative analysis of the dynamics of two mutually destructive reproducing populations

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A R T I C L E I N F O

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A B S T R A C T

In this paper, a parametric linear continuous-time model that describes the dynamics of a conflict between two populations is constructed. It is assumed that individuals of each population destroy individuals of the other population at given rates, and individuals of each population can be reproduced at given rates with new individuals immediately entering into conflict. The paper contains a qualitative (i.e., mathematical) analysis of the proposed model. The relevance of this problem is due to the fact that, in essence, the entire class of similar problems is considered from a single point of view. Besides, the obtained results outline some strong base for developing algorithms (and, consequently, software) for automatic analysis of systems in this class.

1. Introduction

It is well-known that one of the main points for successful solution of applied problems is construction or selection of an adequate mathematical model for the dynamics analysis of the investigated system. At the same time, the system itself, as a rule, is initially described informally, in natural language. It should be noted that the problem of construction or selection of an adequate mathematical model for the dynamics analysis of the investigated system is the most difficult not only for students and post-graduate students, but also for specialists involved in the dynamics analysis of applied systems. Moreover, this problem is becoming especially relevant in connection with research in the field of AI development.

In this paper we present an approach to a unified solution to the problem of constructing or selecting an adequate mathematical model for the dynamics analysis of the investigated system, which ultimately leads to a reduction in the complexity of solving this problem. This approach is based on separating the complexity of the qualitative (i.e., mathematical) analysis of the entire diversity generated by the constructed or selected symbolic (i.e., parametric) mathematical model from the complexity of the quantitative analysis of a specific numerical model selected from this diversity.

A generally accepted approach to solving a wide class of applied problems from various branches of the natural and social sciences (biology, medicine, healthcare, warfare, economics, sports, sociology, etc.) is to study the dynamics of conflicts between two populations, represented via an appropriate mathematical model. This model can often be generalized to an arbitrary number of populations, which makes it possible to expand significantly the range of problems to be solved

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[1, 2]. Apparently, one of the most frequently used mathematical models representing the dynamics of conflicting populations is some version or generalization of the classical nonlinear Lotka-Volterra model [3-5]. It should be noted that along with the continuous-time Lotka-Volterra model determined by differential equations, discrete and probabilistic versions of this model are also successfully used [6, 7].

Along with nonlinear mathematical models, linear mathematical models of conflicts between two populations (as well as their various variants and generalizations) are also successfully used to solve a fairly wide class of applied problems.

Perhaps the first linear mathematical model of the dynamics of conflicts between two populations, which became the basis for the formation of an entire research line, is the basic linear continuous-time Lanchester model [8]. This mathematical model is intended to study the dynamics of combat between two armies that do not replenish their resources. Various variants and generalizations of this mathematical model have numerous applications in solving applied problems from various fields of natural and social sciences [9, 10]. It should also be noted that the discrete Leslie model [11, 12] that has numerous applications in the investigation of age-related population dynamics is some generalization of the discretization of the basic linear continuous-time Lanchester model.

From the above it follows that regardless of what type of mathematical model we choose, the research trends have the same structure, namely:

- The basic model.
- Variants of the basic model.
- Generalization of the basic model and of its variants.

Actions in accordance with this structure make it possible to simplify significantly the construction of mathematical models intended to analyze the dynamics of investigated systems when solving specific applied problems.

In the given paper, taking this statement into account, we will consider the solving of the following problem.

It is necessary to construct or select and investigate a mathematical model intended to analyze the dynamics of two conflicting populations under the following three conditions:

- Individuals of each population destroy individuals of the other population at given rates.
- Individuals of each population are reproduced at given rates.
- New individuals immediately entering into conflict.

This problem has applications in biochemistry, in the analysis dynamics of conflicts between either animal groups, or urban gangs, as well as for the analysis dynamics of confrontation in modern wars between drones and anti-drone weapons, naval drones and warships, tanks and anti-tank weapons, etc.

The structure of the remainder of this paper is as follows. In Section 2 a linear continuous-time symbolic (i.e., parametric) mathematical model designed to solve the investigated problem is constructed. In Section 3 general solution for proposed model is found. In Section 4 the Cauchy problem for proposed model is solved. In Section 5 qualitative analysis for solution of proposed model is presented. Section 6 consists of some concluding remarks.

2. Proposed mathematical model

We could not find any mathematical model adequate to solve the investigated problem. Thus, first of all, we need to construct an adequate mathematical model to solve the investigated problem.

From the standpoint of the investigated problem, the closest mathematical model to it is the basic linear continuous-time Lanchester model [8]:

$$\begin{cases} \dot{A} = -\beta B \\ \dot{B} = -\alpha A \end{cases}$$

where A and B are the numbers of soldiers, respectively, of the Red and Blue armies at instant t, while α and β are positive constants that are the rates of enemy soldiers destruction by one soldier, respectively, of the Red and Blue armies. However, as noted above, in this mathematical model both armies do not replenish their resources. Let us refine this model in order to obtain a mathematical model adequate for solving the investigated problem.

It is assumed that time t is varied in the interval $[0,\infty)$. In accordance with the conditions imposed on the dynamics of two conflicting populations, we need to consider the following quantities:

- $x_i(t)$ $(i=1,2)$ is some non-negative function whose value is equal to the number of individuals of the i-th population at the instant $t \in [0,\infty)$ (it is assumed that each individual takes part in the destruction of individuals of the other population at the instant $t \in [0,\infty)$.
- a_i $(i=1,2)$ is some positive parameter denoting the number of new individuals of the i-th population produced at the instant $t \in [0,\infty)$ per existing individual of this population, and who immediately participate in the destruction of individuals of another population.
- k_i $(i=1,2)$ is some positive parameter denoting the number of individuals of another population, which one element of the i-th population destroys at the instant $t \in [0,\infty)$.

Based on these quantities, we can construct the following autonomous system of linear ordinary differential equations (LODEs) with constant coefficients:

$$\begin{cases} \dot{x}_1(t) = -k_2 x_2(t) + a_1 x_1(t) \\ \dot{x}_2(t) = -k_1 x_1(t) + a_2 x_2(t) \end{cases} \tag{1}$$

Thus, the proposed mathematical model, constructed to solve the investigated problem, consists of the LODEs system (1) with positive parameters k_1, k_2, a_1, a_2 and conditions $x_i(t) \geq 0$ $(i=1,2)$ for all $t \in [0,\infty)$.

It should be noted that if in the LODEs system (1) we formally set $x_i(t) \geq 0$ $(i=1,2)$ for all $t \in [0,\infty)$, $a_i = 0$ $(i=1,2)$, and $k_i \geq 0$ $(i=1,2)$ then we get the basic Lanchester model, while if we set $x_i(t) \geq 0$ $(i=1,2)$ for all $t \in [0,\infty)$, $a_1 \geq 0$, $a_2 = 0$, $k_1 \in (0,1)$, and $k_2 < 0$ then upon discretization of the resulting LODEs system we get the standard 2×2-Leslie model.

It should be noted that LODEs systems, quite close to (1), have been considered in [13,14]. However, the interpretation of the quantities included in them has been completely different.

Representing LODEs system (1) in the matrix form, we get

$$\dot{x}(t) = Ax(t) \tag{2}$$

where

$$A = \begin{pmatrix} a_1 & -k_2 \\ -k_1 & a_2 \end{pmatrix} \tag{3}$$

and $x(t) = (x_1(t), x_2(t))^T$.

3. General solution for proposed mathematical model

Let us recall that the proposed mathematical model, constructed to solve the investigated problem, consists of the LODEs system (1) with positive parameters k_1, k_2, a_1, a_2 and conditions $x_i(t) \geq 0$ $(i=1,2)$ for all $t \in [0,\infty)$.

3.1 The Eigenvalues of the Matrix A

The eigenvalues of the 2×2-matrix A are the roots of the equation

$$\det(A-rI_2)=0,$$

where I_2 is the unit 2×2-matrix.

Let us find the roots of this equation for the matrix A given by formula (3). Let us represent this equation in the explicit form:

$$\det(A-rI_2)=0 \Leftrightarrow (a_1-r)(a_2-r)-k_1k_2=0 \Leftrightarrow$$
$$\Leftrightarrow r^2-(a_1+a_2)r+(a_1a_2-k_1k_2)=0.$$

Since

$$D=(a_1+a_2)^2-4(a_1a_2-k_1k_2)=(a_1-a_2)^2+4k_1k_2>0, \quad (4)$$

the matrix A always has two different real eigenvalues

$$r_1=0.5(a_1+a_2)+\sqrt{D}, \quad (5)$$
$$r_2=0.5(a_1+a_2)-\sqrt{D}. \quad (6)$$

Thus, the general solution for the proposed model consists of non-negative values of a two-parameter family of functions

$$x(t)=c_1e^{r_1t}v_1+c_2e^{r_2t}v_2, \quad (7)$$

where c_1 and c_2 are parameters, i.e. arbitrary constants, and $v_i=(v_1^{(i)},v_2^{(i)})^T$ $(i=1,2)$ is any eigenvector for the matrix A associated with the eigenvalue r_i.

Now we establish the basic properties of the eigenvalues r_i $(i=1,2)$, which we will need later. From (4)-(6) it follows directly that the following three propositions are true.

Proposition 1. For all admissible values of parameters k_1, k_2, a_1, a_2 inequalities $r_1>0$ and $r_1>r_2$ are true.

Proposition 2. For all admissible values of parameters k_1, k_2, a_1, a_2 formula

$$r_2\begin{cases}<0, & if\ a_1a_2<k_1k_2\\ =0, & if\ a_1a_2=k_1k_2\\ >0, & if\ a_1a_2<k_1k_2\end{cases}$$

is true.

Proposition 3. For all admissible values of parameters k_1, k_2, a_1, a_2 inequalities $a_j-r_1<0$ and $a_j-r_2>0$ are true for all $j\in\{1.2\}$.

3.2 Eigenvectors of the Matrix A

It is known that any eigenvector v of the 2×2-matrix A associated with an eigenvalue r is a non-zero root of the equation

$$Av=rv \Leftrightarrow (A-rI_2)v=0_2,$$

where $0_2=(0,0)^T$. Thus, for the matrix A defined by formula (3), any eigenvector $v_i=(v_1^{(i)},v_2^{(i)})^T$ $(i=1,2)$ associated with the eigenvalue r_i is a non-zero solution to the system of equations

$$\begin{cases}(a_1-r_i)v_1^{(i)}-k_2v_2^{(i)}=0\\ -k_1v_1^{(i)}+(a_2-r_i)v_2^{(i)}=0\end{cases} \quad (i=1,2). \quad (8)$$

To find eigenvectors $v_i=(v_1^{(i)},v_2^{(i)})^T$ $(i=1,2)$ we need the following theorem.

Theorem. For all admissible values of parameters k_1, k_2, a_1, a_2 both components of any eigenvector $v_i = (v_1^{(i)}, v_2^{(i)})^T$ $(i = 1,2)$ associated with the eigenvalue r_i are nonzero.

Proof. Let $i = 1$.

Assume the opposite, i.e. that there exists an eigenvector $v_1 = (v_1^{(1)}, v_2^{(1)})^T$ with a zero component associated with the eigenvalue r_1.

Let's assume that $v_1^{(1)} = 0$. From the first equation of system (8) it follows that $k_2 v_2^{(1)} = 0$. Since $k_2 > 0$ then $v_2^{(1)} = 0$. Thus, $v_1 = (0,0)^T$, which contradicts the definition of an eigenvector. Therefore, $v_1^{(1)} \neq 0$.

Let's assume that $v_2^{(1)} = 0$. From the first equation of system (8) it follows that $(a_1 - r_1)v_1^{(1)} = 0$. From Proposition 3 it follows that $a_1 - r_1 \neq 0$. Thus, $v_1^{(1)} = 0$, and $v_1 = (0,0)^T$, which contradicts the definition of an eigenvector. Therefore, $v_2^{(1)} \neq 0$.

For $i = 2$ proof is carried out similarly, using the second equation of system (8).

Now we can find eigenvectors $v_i = (v_1^{(i)}, v_2^{(i)})^T$ $(i = 1,2)$.

It follows from Theorem 1 that for both eigenvalues r_i $(i = 1,2)$ both components of any eigenvector $v_i = (v_1^{(i)}, v_2^{(i)})^T$ are nonzero. Therefore, without loss of generality, we will assume that $v_1^{(i)} = 1$ $(i = 1,2)$. From the first equation of system (8) we get

$$a_1 - r_i - k_2 v_2^{(i)} = 0 \Leftrightarrow v_2^{(i)} = (a_1 - r_i)k_2^{-1} \quad (i = 1,2).$$

Thus,

$$v_i = (1, (a_1 - r_i)k_2^{-1})^T \quad (i = 1,2). \tag{9}$$

From Proposition 3 it follows that $(a_1 - r_1)k_2^{-1} < 0$ and $(a_1 - r_2)k_2^{-1} > 0$. Thus, from (9) it follows that the eigenvector v_1 lies in the 4-th quadrant, and the eigenvector v_2 lies in the 1-st quadrant.

3.3 Formula of General Solution for Proposed Mathematical Model

Substituting (9) into (7), we get

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{r_1 t} \begin{pmatrix} 1 \\ (a_1 - r_1)k_2^{-1} \end{pmatrix} + c_2 e^{r_2 t} \begin{pmatrix} 1 \\ (a_1 - r_2)k_2^{-1} \end{pmatrix} \tag{10}$$

Substituting (5) and (6) into (10), we get the general solution for the proposed model, which consists of non-negative values of a two-parameter family of functions

$$\begin{cases} x_1(t) = e^{0,5(a_1+a_2)t}(c_1 e^{0.5t\sqrt{D}} + c_2 e^{-0.5t\sqrt{D}}) \\ x_2(t) = 0.5k_2^{-1} e^{0.5(a_1+a_2)t}(c_1 e^{0.5t\sqrt{D}}(a_1 - a_2 - \sqrt{D}) + c_2 e^{-0.5t\sqrt{D}}(a_1 - a_2 + \sqrt{D})) \end{cases} \tag{11}$$

where D is defined by formula (4), c_1 and c_2 are parameters, i.e. arbitrary constants.

4. Cauchy problem for proposed mathematical model

In our case, Cauchy problem consists in finding non-negative pieces of the solution of the LODEs system (1) that satisfy the given admissible initial conditions for the given admissible values of proposed mathematical model parameters k_1, k_2, a_1, a_2.

4.1 Analysis of Admissible Parameters c_1 and c_2 Values

Let

$$\begin{cases} x_1(0) = x_1^{in} \\ x_2(0) = x_2^{in} \end{cases} \tag{12}$$

be the initial conditions, such that $x_1^{in} \geq 0$ and $x_2^{in} \geq 0$.

Let's find the associated values of the parameters c_1 and c_2.

Setting $t = 0$ into (11) and substituting (12), we get

$$\begin{cases} x_1^{in} = c_1 + c_2 \\ x_2^{in} = 0.5k_2^{-1}(c_1(a_1 - a_2 - \sqrt{D}) + c_2(a_1 - a_2 + \sqrt{D})) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1^{in} = c_1 + c_2 \\ x_2^{in} = 0.5k_2^{-1}((a_1 - a_2)(c_1 + c_2) - \sqrt{D}(c_1 - c_2)) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1^{in} = c_1 + c_2 \\ x_2^{in} = 0.5k_2^{-1}((a_1 - a_2)x_1^{in} - \sqrt{D}(c_1 - c_2)) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c_1 + c_2 = x_1^{in} \\ c_1 - c_2 = \dfrac{1}{\sqrt{D}}((a_1 - a_2)x_1^{in} - 2k_2 x_2^{in}) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c_1 = 0.5((1 + \dfrac{1}{\sqrt{D}}(a_1 - a_2))x_1^{in} - \dfrac{2k_2}{\sqrt{D}} x_2^{in}) \\ c_2 = 0.5((1 - \dfrac{1}{\sqrt{D}}(a_1 - a_2))x_1^{in} + \dfrac{2k_2}{\sqrt{D}} x_2^{in}) \end{cases} \tag{13}$$

Now we analyze the signs of parameters c_1 and c_2 depending on the relationships between admissible proposed model parameters k_1, k_2, a_1, a_2 values, and initial values of x_1^{in} and x_2^{in}.

Let $\rho_1, \rho_2 \in \{<, >, =\}$. From (13) it follows that

$$\begin{cases} c_1 \rho_1 0 \\ c_2 \rho_2 0 \end{cases} \Leftrightarrow \begin{cases} 0.5((1 + \dfrac{1}{\sqrt{D}}(a_1 - a_2))x_1^{in} - \dfrac{2k_2}{\sqrt{D}} x_2^{in})\rho_1 0 \\ 0.5((1 - \dfrac{1}{\sqrt{D}}(a_1 - a_2))x_1^{in} + \dfrac{2k_2}{\sqrt{D}} x_2^{in})\rho_2 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} \rho_1 x_2^{in} \\ x_2^{in} \rho_2 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \tag{14}$$

We note that from (4) it follows that $\sqrt{D} > |a_1 - a_2|$. Therefore,

$$0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} \geq 0 \tag{15}$$

and

$$0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \leq 0 \tag{16}$$

for all admissible proposed model parameters k_1, k_2, a_1, a_2 values, and initial values of x_1^{in} and x_2^{in}.

4.2 Analysis of Admissible Combinations of Parameters c_1 and c_2 Signs

Now using (14) we will examine all possible combinations of parameters c_1 and c_2 signs.

Case 1. Let $c_1 < 0$ and $c_2 < 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} < x_2^{in} \\ x_2^{in} < 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \tag{17}$$

From (15) and (16) it follows that formula (17) is false for all admissible proposed model parameters k_1, k_2, a_1, a_2 values, and initial values of x_1^{in} and x_2^{in}. Thus, Case 1 is impossible when we are solving considered Cauchy problem.

Case 2. Let $c_1 < 0$ and $c_2 > 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} < x_2^{in} \\ x_2^{in} > 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \Leftrightarrow x_2^{in} > 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in}.$$

Thus, Case 2 occurs for all admissible proposed model parameters k_1, k_2, a_1, a_2 values and initial values such that $x_1^{in} \geq 0$ and $x_2^{in} \in (0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in}, \infty)$.

Case 3. Let $c_1 < 0$ and $c_2 = 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} < x_2^{in} \\ x_2^{in} = 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \tag{18}$$

From (15) and (16) it follows that formula (18) is false for all admissible proposed model parameters k_1, k_2, a_1, a_2 values and initial values of x_1^{in} and x_2^{in}. Thus, Case 3 is impossible when we are solving considered Cauchy problem.

Case 4. Let $c_1 > 0$ and $c_2 < 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} > x_2^{in} \\ x_2^{in} < 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \Leftrightarrow x_2^{in} < 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in}. \tag{19}$$

From (16) it follows that formula (19) is false for all admissible proposed model parameters k_1, k_2, a_1, a_2 values, and initial values of x_1^{in} and x_2^{in}. Thus, Case 4 is impossible when we are solving considered Cauchy problem.

Case 5. Let $c_1 > 0$ and $c_2 > 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} > x_2^{in} \\ x_2^{in} > 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} < x_2^{in} < 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in}.$$

Taking formula (16) into account, we conclude that Case 5 occurs for all admissible proposed model parameters k_1, k_2, a_1, a_2 values and all initial values such that $x_1^{in} > 0$ and $x_2^{in} \in (0, 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in})$.

Case 6. Let $c_1 > 0$ and $c_2 = 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} > x_2^{in} \\ x_2^{in} = 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \tag{20}$$

From (15) and (16) it follows that formula (20) is false for all admissible proposed model parameters k_1, k_2, a_1, a_2 values, and initial values of x_1^{in} and x_2^{in}. Thus, Case 6 is impossible when we are solving considered Cauchy problem.

Case 7. Let $c_1 = 0$ and $c_2 < 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} = x_2^{in} \\ x_2^{in} < 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases} \quad (21)$$

From (15) and (16) it follows that formula (21) is false for all admissible proposed model parameters k_1, k_2, a_1, a_2 values, and initial values of x_1^{in} and x_2^{in}. Thus, Case 7 is impossible when we are solving considered Cauchy problem.

Case 8. Let $c_1 = 0$ and $c_2 > 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} = x_2^{in} \\ x_2^{in} > 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases}$$

From (15) and (16) it follows that Case 8 occurs for all admissible proposed model parameters k_1, k_2, a_1, a_2 values and all initial values such that $x_1^{in} > 0$ and $x_2^{in} = 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in}$.

Case 9. Let $c_1 = 0$ and $c_2 = 0$. We get

$$\begin{cases} 0.5k_2^{-1}(a_1 - a_2 + \sqrt{D})x_1^{in} = x_2^{in} \\ x_2^{in} = 0.5k_2^{-1}(a_1 - a_2 - \sqrt{D})x_1^{in} \end{cases}$$

From (15) and (16) it follows that Case 9 occurs for all admissible proposed model parameters k_1, k_2, a_1, a_2 values and initial values $x_1^{in} = 0$ and $x_2^{in} = 0$.

5. Qualitative analysis for solution of proposed model

It was established above that when studying the proposed mathematical model, only parts of solutions that satisfy conditions $x_1^{in} \geq 0$ and $x_2^{in} \geq 0$ are of interest. Based on these conditions, it follows from (11) that the admissible time values can be found from the inequalities

$$\begin{cases} c_1 e^{t\sqrt{D}} + c_2 \geq 0 \\ c_1 e^{t\sqrt{D}}(a_1 - a_2 - \sqrt{D}) + c_2(a_1 - a_2 + \sqrt{D})) \geq 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c_1 e^{t\sqrt{D}} \geq -c_2 \\ c_1 e^{t\sqrt{D}} \leq c_2 \dfrac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \end{cases} \quad (22)$$

Now we will find the admissible time values for each of the above-established possible cases 2, 5, 8 and 9 for Cauchy problem.

For case 2 the conditions $c_1 < 0$ and $c_2 > 0$ are met. From (22), we get

$$\begin{cases} e^{t\sqrt{D}} \leq -c_1^{-1}c_2 \\ e^{t\sqrt{D}} \geq c_1^{-1}c_2 \dfrac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \end{cases} \Leftrightarrow \begin{cases} t \leq \dfrac{1}{\sqrt{D}} \ln(-c_1^{-1}c_2) \\ t \in [0, \infty) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow 0 \le t \le \frac{1}{\sqrt{D}} \ln(-c_1^{-1} c_2) . \tag{23}$$

From (23) it follows that inequality $-c_1^{-1} c_2 \ge 1$ must be true. Moreover, having designated $L = \frac{1}{\sqrt{D}} \ln(-c_1^{-1} c_2)$, based on formula (11) and performing algebraic transformations, we obtain

$$\lim_{t \to L} x_1(t) = 0$$

and

$$\lim_{t \to L} x_2(t) = c_2 k_2^{-1} \sqrt{D} (-c_1^{-1} c_2)^{\frac{a_1 + a_2}{2\sqrt{D}} - 0.5} .$$

For case 5 the conditions $c_1 > 0$ and $c_2 > 0$ are met. From (22), we get

$$\begin{cases} e^{t\sqrt{D}} \ge -c_1^{-1} c_2 \\ e^{t\sqrt{D}} \le c_1^{-1} c_2 \dfrac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \end{cases} \Leftrightarrow \begin{cases} t \in [0, \infty) \\ t \le \dfrac{1}{\sqrt{D}} \ln\left(c_1^{-1} c_2 \dfrac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \right) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow 0 \le t \le \frac{1}{\sqrt{D}} \ln\left(c_1^{-1} c_2 \frac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \right).$$

From (26) it follows that inequality $c_1^{-1} c_2 \frac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \ge 1$ must be true. Moreover, having designated $L = \frac{1}{\sqrt{D}} \ln\left(c_1^{-1} c_2 \frac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \right)$ based on formula (11) and performing algebraic transformations, we obtain

$$\lim_{t \to L} x_1(t) = \frac{\sqrt{c_1 c_2 D}}{\sqrt{a_1 a_2 + k_1 k_2}} \left(c_1^{-1} c_2 \frac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \right)^{\frac{a_1 + a_2}{2\sqrt{D}}}$$

and

$$\lim_{t \to L} x_2(t) = 0 .$$

For case 8 the conditions $c_1 = 0$ and $c_2 > 0$ are met. From (22), we get

$$\begin{cases} c_2 \ge 0 \\ c_2 \dfrac{a_1 - a_2 + \sqrt{D}}{\sqrt{D} - (a_1 - a_2)} \ge 0 \end{cases} \tag{24}$$

Obviously, formula (24) is true for all $t \in [0, \infty)$. Based on formula (11) and performing algebraic transformations, we obtain

$$\lim_{t \to \infty} x_1(t) = \begin{cases} 0, \ if \ \ a_1 a_2 < k_1 k_2 \\ c_2, \ if \ \ a_1 a_2 = k_1 k_2 \\ \infty, \ if \ \ a_1 a_2 > k_1 k_2 \end{cases}$$

and

$$\lim_{t\to\infty} x_2(t) = \begin{cases} 0, \ if \ \ a_1 a_2 < k_1 k_2 \\ 0.5 k_2^{-1} c_2 (a_1 - a_2 + \sqrt{D}), \ if \ \ a_1 a_2 = k_1 k_2 \\ \infty, \ if \ \ a_1 a_2 > k_1 k_2 \end{cases}$$

For case 9 the conditions $c_1 = 0$ and $c_2 > 0$ are met. From (11) it follows that $x_1(t) = 0$ and $x_2(t) = 0$ for all $t \in [0, \infty)$.

6. Conclusion

The main objective of the given paper is to present step-by-step the following two components of the process of solving the problem of analyzing any investigated system dynamics.

The first component is to choose or to construct an adequate mathematical model intended to solve this problem. High complexity of this component, among other things, is often due to the fact that the investigated dynamic system is described informally, in natural language. It is well-known that this component is decisive for the successful solving of the considered problem. At the same time, this component causes great difficulty both for students and postgraduates, as well as for a significant number of researchers engaged in solving this problem.

The second component is to present step-by-step the mathematical analysis of the study of (selected or constructed) parametric mathematical model intended to solve the considered problem. The emphasis on choosing a parametric mathematical model is due to the fact that it makes it possible to simultaneously solve an entire class of similar problems. Moreover, it is the parametric mathematical model that makes it possible to construct its natural generalizations, intended for refining and generalizing possible further studies of the considered problem.

In the given paper the investigated dynamic system is a system of two populations capable of reproduction and striving to destroy each other, with new individuals immediately participating in this process. To solve the problem of analyzing the dynamics of this system, it was decided to limit ourselves to a linear mathematical model presented in parametric form. Since an adequate linear mathematical model intended to solve the considered problem was not found, it was decided to modify the closest classical Lanchester model. This step is presented in detail in the given paper. A step-by-step qualitative (i.e. mathematical) analysis of the constructed parametric linear mathematical model is presented. This qualitative analysis, among other things, defines the following structure of software intended for computer simulation of numerical variants of proposed linear mathematical model with the possibility of stopping after each step of calculations:

Step 1. The specified admissible parameter values are entered.

Step 2. The eigenvalues of the matrix A are calculated.

Step 3. The eigenvectors of the matrix A are calculated.

Step 4. The general solution of the system of LODEs is calculated.

Step 5. Admissible initial conditions x_1^{in} and x_2^{in} are entered.

Step 6. The values of parameters c_1 and c_2 are calculated.

Step 7. Admissible intervals for variables and time are calculated.

Step 8. The limits for the values of variables when time tends to the right end of the interval are calculated.

The trends for further possible research are as follows.

Firstly, this is the development of a software system intended for computer simulation of the results presented in the given paper.

Secondly, this is a qualitative analysis of generalizations of the parametric mathematical model constructed in the given paper for cases where the parameters are functions of time that have significant substantive value, as well as the development of appropriate software intended for computer simulation of numerical variants of these mathematical models.

Thirdly, this is construction of parametric stochastic analogues of the above pointed mathematical models, their qualitative analysis, and development of appropriate software.

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